Math 113. Sample Final Exam

Solve 6 of the following 10 problems.

1. How many cyclic subgroups are there in the group $\mathbb{Z}_5^3$?

2. How many different group isomorphisms $\mathbb{Z}_{120} \to \mathbb{Z}_{120}$ are there?

3. Is the group $\mathbb{Z}_{1000}^{\times}$ (of units of the ring $\mathbb{Z}_{1000}$) cyclic? Why?

4. Let I be the principal ideal in $\mathbb{Z}_5[x]$ generated by the polynomial $x^3 + x^2 + x + 3$. Prove that $\mathbb{Z}_5[x]/I$ is a field and find the number of elements in it.

5. How many roots does the polynomial $x^{100000} - 1$ have in the field $\mathbb{Z}_{65537}$, where $65537 = 2^{2^4} + 1$ is the fourth Fermat prime number F_4?

6. How many permutations of order 3 are there in the group S_6 of permutations of 6 objects?

7. Let I be the ideal in $\mathbb{R}[x]$ generated by the polynomial $x^8 + x^2 + 2$. How many maximal ideals are in the quotient ring $\mathbb{R}[x]/I$? Why?

8. Find the greatest common divisor of $f = x^7 + 6x^3 - 4x - 12$ and $g = x^5 - 3x^4 + 2x^2 - 18$, and determine whether it is irreducible in $\mathbb{Q}[x]$.

9. A *benzene* molecule C_6H_6 has the shape of a regular hexagon formed by 6 carbon atoms with one hydrogen atom attached to each of them. There are two stable isotopes of carbon: ^{12}C and ^{13}C, and two of hydrogen: ^{1}H and ^{2}H. How many stereo-isotopes (i.e. molecules geometrically different due to presence of isotopes) of benzene can be found?

10. Let $t := e^{2\pi i/5} + e^{-2\pi i/5} = 2\cos 2\pi/5$. Find the minimal degree polynomial in $\mathbb{Q}[x]$ which has t as its root, and derive that regular pentagons can be constructed by straightedge and compass.