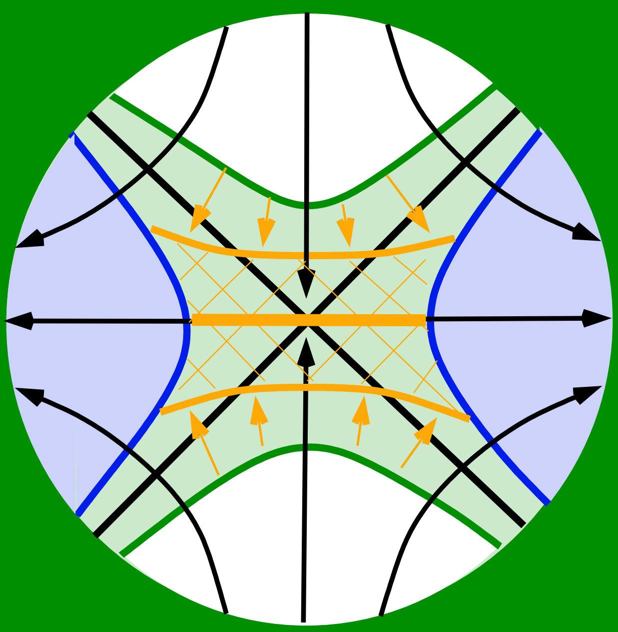
Lecture notes on Algebraic Topology



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Lecture notes

on

Algebraic Topology

by

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Preface

This book is a derivative product.

It represents a one-semester graduate-level course taught by A.G. at UC Berkeley based on Chapters 1 and 2 of the textbook [2] by D.F. as well as on A.G.'s own student notes in the courses taught by D.F. at the Moscow State University in 1976-78.

It should not therefore be surprising that here homology and cohomology, in agreement with [2] (and in contrast with the popular textbook [8] by A. Hatcher), come only after the basics of homotopy theory are developed, the former being often treated merely as efficient tools for handling problems of the latter. To a reader who is content with this paradigm we would still recommend the far more substantial original [2], but suspect that for an intensive one-semester or less intensive two-quarter course in algebraic topology, a student and instructor might find the present more selective and concise exposition also useful.

Beside its scope and size, our presentation deviates from [2] in some other ways. We are less concerned with the needs of algebraic topology *per se* than with potential applications of it in more general geometric contexts. For instance, instead of piecewise linear techniques of [2], we resort here to smooth approximations. Respectively, we assume some basic familiarity with analysis on manifolds (Riemannian metrics, gradient flows), quote Sard's lemma on several important occasions, and follow a Morse-theoretic approach to Poincaré duality and intersection theory.

Also, we refrain from delegating any essential aspects of proofs to exercises, and honestly hope that our misprints (for which the authors of [2], obviously, bear no responsibility) are complementary to theirs.

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Lecture 1. Dramatis personae

We begin with some examples of topological spaces which, as we will see in the future, play key roles in our theory.

A. Vector spaces, disks, spheres. We use the notation $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ for the coordinate n-dimensional real, complex, and quaternionic vector spaces.

The subset in $\mathbb{R}^n$

$$D^n := \{(x_1, \dots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

is called the n-dimensional *disk*, and its boundary

$$\partial D^n = S^{n-1} = \{(x_1, \dots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1\}$$

is the $(n-1)$-dimensional *sphere*.

Consider the nested sequence $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \dots \mathbb{R}^k \subset \dots$ of vector spaces. The union

$$\mathbb{R}^\infty = \{(x_1, x_2, \dots, x_k, \dots) \mid \text{all but finitely many } x_k = 0\ \}$$

is the version of infinite dimensional vector spaces we will usually need. It is equipped with the topology of *direct limit*: A subset in $\mathbb{R}^\infty$ is closed (open) if and only if its intersection with each $\mathbb{R}^k$ is closed (resp. open). One can similarly define $\mathbb{C}^\infty, \mathbb{H}^\infty$ as well as $S^\infty \subset D^\infty \subset \mathbb{R}^\infty$.

B. Classical groups. The *orthogonal group* O_n is defined as the group of linear transformations in $\mathbb{R}^n$ preserving the standard *Euclidean inner product* $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{k=1}^n x_k y_k$. It consists of real $n \times n$-matrices U satisfying $U^t U = I$ (here I is the identity $n \times n$-matrix, and "t" denotes transposition), and inherits the topology from the ambient space $\mathbb{R}^{n^2}$ of all $n \times n$-matrices. The space O_n has two connected components formed by the matrices with determinants ± 1. Those with $\det U = 1$ form the *special orthogonal group* SO_n.

The *unitary group* U_n is similarly defined as the group of linear transformations of $\mathbb{C}^n$ preserving the standard *Hermitian inner product* $\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{k=1}^n z_k \bar{w}_k$. It consists of complex $n \times n$-matrices satisfying $\overline{U}^t U = I$. The kernel of the group homomorphism $\det : U_n \to U_1 = \{u \in \mathbb{C} \mid |u| = 1\}$ is called the *special unitary group* and denoted SU_n.

We have: $O_1 = \{\pm 1\} \simeq S^0$, $SO_2 \cong U_1 \simeq S^1$ (where "$\simeq$" stands for "homeomorphic").

The quaternionic version of the orthogonal and unitary groups is called the *compact symplectic group* and denoted Sp_n. To introduce it properly, let us first recall what quaternions are.

C. Quaternions. By definition,

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

is an associative $\mathbb{R}$-algebra with the basis $1, i, j, k$ obeying the relations $i^2 = j^2 = k^2 = ijk = -1$. The relations imply $k = ij = -ji$, which allows one to rewrite the quaternions in complex notation: $\mathbb{H} = \{z + wj \mid z, w \in \mathbb{C}\}$, where $j^2 = -1$, and $z = a + bi$, $w = c + di$. The quaternion conjugated to $q = z + wj$ is introduced as $q^* = a - bi - cj - dk = \bar{z} - wj$, and one can easily check that

$$q^* q = q q^* = |z|^2 + |w|^2 = a^2 + b^2 + c^2 + d^2 =: \|q\|^2.$$

This makes $\mathbb{H}$ a *division algebra*: $1/q = q^*/\|q\|^2$ is well-defined for all $q \neq 0$. This shows that the Gaussian elimination algorithm, and along with it all basic linear algebra carry over to vector spaces over $\mathbb{H}$ in the role of scalars. One only needs to remember that if multiplication in $\mathbb{H}^n$ by quaternionic scalars acts on the left: $\mathbf{q} \mapsto \lambda \mathbf{q}$, then an $\mathbb{H}$-linear transformation of $\mathbb{H}^n$ is described as the multiplication of a row $\mathbf{q} = (q_1, \ldots, q_n)$ by a quaternionic $n \times n$-matrix on the right.

By definition, the group Sp_n consists of such linear transformations preserving the *Hamiltonian inner product* $\langle \mathbf{q}', \mathbf{q} \rangle := \sum_{k=1}^n q'_k q^*_k$. Note that it is $\mathbb{H}$-valued, and is $\mathbb{H}$-linear in $\mathbf{q}'$ and anti-linear in $\mathbf{q}$: $\langle \lambda \mathbf{q}', \mu \mathbf{q} \rangle = \lambda \langle \mathbf{q}', \mathbf{q} \rangle \mu^*$.

Consider now the case $n = 1$ and examine the right multiplication of $q = (x + yj)$ by $\mu = z + wj$ on the right, taking into account that j anti-commutes with i:

$$(x + yj)(z + wj) = (xz - y\bar{w}) + (xw + y\bar{z})j.$$

This can be interpreted as the multiplication of a row $(x, y) \in \mathbb{C}^2$ on the right by the complex matrix $\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$. (By the way, this establishes the existence of $\mathbb{H}$.) Note that the rows of this 2×2-complex matrix are Hermitian orthogonal. The group Sp_1 consists of such matrices with $\|z + wj\|^2 = |z|^2 + |w|^2 = 1$. The last relation has triple meaning: It shows that the two rows of the matrix are unit in the Hermitian sense, and also that its determinant equals 1, i.e. that such matrices form the special unitary group SU_2. We conclude that $SU_2 \cong Sp_1 \simeq S^3$, the 3-sphere of unit quaternions.

Consider now the action of Sp_1 on $\mathbb{H} = \mathbb{R}^4$ by conjugations. They preserve the lengths of all vectors in $\mathbb{R}^4$, hence preserve the Euclidean inner product in $\mathbb{R}^4$, and since they commute with $1 \in \mathbb{H}$, they preserve the orthogonal to 1 subspace $\mathbb{R}^3 = \{bi + cj + dk\}$ of imaginary quaternions. Thus, we obtain a homomorphism from the connected group Sp_1 to SO_3. It is not hard to see that the homomorphism is surjective, and its kernel consists of ± 1. (E.g., one can argue that this kernel must lie in the center of $\mathbb{H}$, which is the real axis. Since the kernel is discrete, and both groups are 3-dimensional, the map $Sp_1 \to SO_3$ is a local diffeomorphism near the identity, and then is surjective, since the rank of a smooth homomorphism between Lie groups must be constant.) Therefore, $SO_3 \cong SU_2 / \pm I$.

D. Projective spaces. Let $\mathbb{K}$ denote one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. The *projective space* $\mathbb{K}P^n$ is defined as the set of 1-dimensional linear subspaces in $\mathbb{K}^{n+1}$. One way of thinking of it is shown in Figure 1:

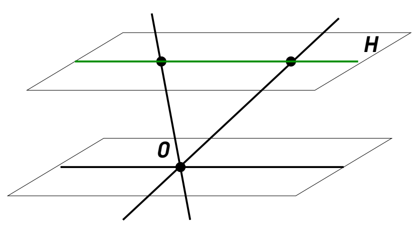

Figure 1: $\mathbb{K}P^n = \mathbb{K}^n \cup \mathbb{K}P^{n-1}$

Points of a fixed affine hyperplane H in $\mathbb{K}^{n+1}$ correspond to all 1-dimensional subspaces except those which are parallel it. The latter form the projective space $\mathbb{K}P^{n-1}$ of "points of H at infinity": A line in H (shown green) intersects this $\mathbb{K}P^{n-1}$ at one point, the same one for all lines parallel to it. In particular, $\mathbb{K}P^1 = \mathbb{K} \cup (\infty)$: a line $y = kx$ in $\mathbb{K}^2$ has slope $k \in \mathbb{K}$ unless $k = \infty$. Therefore $\mathbb{R}P^1 \simeq S^1$, $\mathbb{C}P^1 \simeq S^2$ (the Riemann sphere), $\mathbb{H}P^1 \simeq S^4$.

Alternatively (Figure 2), $\mathbb{R}P^n = S^n/(\pm 1)$: Every 1-dimensional subspace in $\mathbb{R}^{n+1}$ intersects the unit sphere centered at the origin at two diametrically opposite points $\pm x$.

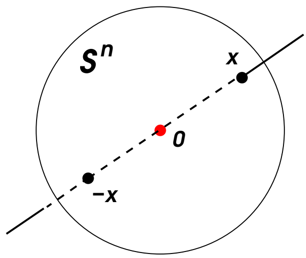

Figure 2: $\mathbb{R}P^n = S^n/O_1$

Similarly, $\mathbb{C}P^n = S^{2n+1}/U_1$ where $U_1 \simeq S^1$ acts freely on the sphere of unit vectors in $\mathbb{C}^{n+1}$ as scalar multiplication by $e^{i\theta}$. The projection $S^{2n+1} \stackrel{S^1}{\to} \mathbb{C}P^n$ is called the *Hopf fibration* or *Hopf bundle*. The quaternionic version of the Hopf fibration is $S^{4n+3} \stackrel{S^3}{\to} \mathbb{H}P^n$, where the fibers S^3 are the orbits of Sp_1 acting on the set S^{4n+3} of unit vectors in $\mathbb{H}^{n+1}$.

By the way, such quotient representation of projective spaces defines their topology. Namely, for any topological space X and an equivalence relation $\sim$ on it, the *quotient topology* on the set $X/\sim$ of equivalence classes is defined as the strongest topology which makes the canonical projection $\pi : X \to X/\sim$ continuous, i.e. $U \subset X/\sim$ is open if and only if $\pi^{-1}(U)$ is open in X.

Note the terminology: A topology with more open sets is *weaker*, so that the discrete topology (the default topology on any set) is the weakest of all.

Using direct limits (or starting from $\mathbb{K}^\infty$ right away) one defines $\mathbb{K}P^\infty$, together with the fibrations

$$S^\infty \stackrel{S^0}{\to} \mathbb{R}P^\infty, \quad S^\infty \stackrel{S^1}{\to} \mathbb{C}P^\infty, \quad S^\infty \stackrel{S^3}{\to} \mathbb{H}P^\infty.$$

E. Stiefel manifolds. The space of all orthonormal *k-frames* in $\mathbb{R}^n$, i.e. k-tuples $(v_1, \dots, v_k)$ of unit pairwise orthogonal vectors in the Euclidean n-space is called a *Stiefel manifold* and is denoted $V(n, k)$. Obviously, $V(n, n) \simeq O_n$. On the other hand, $V(n, k) = O_n/O_{n-k}$. Indeed, a k-frame can be completed to an orthonormal basis, and all such completions $(v_{k+1}, \dots, v_n)$ differ from each other by orthogonal transformations in the orthogonal complement to $Span(v_1, \dots, v_n)$.

So, we have two ways of describing the topology of a Stiefel manifold: the induced topology of a subset in the space $\mathbb{R}^{nk}$ of k-tuples of vectors in $\mathbb{R}^n$, and the quotient topology of O_n/O_{n-k}. We leave it as an exercise for the reader to explain why the two topologies coincide.

The complex versions $\mathbb{C}V(n,k) \simeq U_n/U_{n-k}$ and quaternionic versions $\mathbb{H}V(n,k) \simeq Sp_n/Sp_{n-k}$ of Stiefel manifolds are defined similarly as spaces of k-frames in $\mathbb{C}^n$ and $\mathbb{H}^n$ orthonormal with respect to the Hermitian and Hamiltonian inner products respectively. One can also consider orthonormal k-frames in $\mathbb{K}^\infty$ and thus define $V(\infty,k)$, $\mathbb{C}V(\infty,k)$, and $\mathbb{H}V(\infty,k)$.

Note that for $k=1$ the Stiefel manifolds are spheres of appropriate (possibly infinite) dimensions.

F. Grassmannians. A *grassmannian* or *Grassmann manifold* $G(n,k)$ is the space of all k-dimensional subspaces in $\mathbb{R}^n$. It is topologized by its identification with the quotient space $V(n,k)/O_k = O_n/(O_k \times O_{n-k})$: Orthonormal bases in the same subspace are k-frames in $\mathbb{R}^n$ from the same O_k-orbit. Passing to the orthogonal complement of a subspace one identifies $G(n,k)$ with $G(n,n-k)$. The complex and quaternionic grassmannians $\mathbb{C}G(n,k)$ and $\mathbb{H}G(n,k)$ are defined similarly and are similarly expressed as quotients of U_n and Sp_n respectively. However, in the real case we also have the grassmannian $G_+(n,k)$ of *oriented* k-dimensional subspaces in $\mathbb{R}^n$. Recall that two bases in a real vector space are said to define the same *orientation* if the determinant of the transition matrix between the bases is positive (and define opposite orientations if it is negative). Thus, forgetting orientations defines a 2-to-1 map $G_+(n,k) \to G(n,k)$. When $k=n$ or $k=0$, the "oriented" grassmannian consists of two points (by definition, a 0-dimensional real space has two orientations: $+$ and $-$), but for $0<k<n$, $G_+(n,k) = G_+(n,n-k) = SO_n/(SO_k \times SO_{n-k})$ is connected.

The nested sequence of subspaces

$$\mathbb{K}^k \subset \cdots \subset \mathbb{K}^n \subset \mathbb{K}^{n+1} \subset \ldots$$

defines the tower of embeddings (in the real case the prefix "$\mathbb{R}$" should be omitted):

$$\mathbb{K}G(k,k) \subset \cdots \subset \mathbb{K}G(n,k) \subset \mathbb{K}G(n+1,k) \subset \ldots,$$

where a k-dimensional subspace in $\mathbb{K}^n$ is considered a k-dimensional subspace in $\mathbb{K}^{n+1}$. Passing to the direct limit we obtain the infinite dimensional grassmannian $\mathbb{K}G(\infty,k)$ of k-dimensional subspaces in $\mathbb{K}^\infty$ (and similarly $G_+(\infty,k)$ in the real case).

The inclusion $\mathbb{K}^n \subset \mathbb{K}^{n+1} = \mathbb{K}^n \oplus \mathbb{K}^1$ induces another canonical embedding: $\mathbb{K}G(n,k) \subset \mathbb{K}G(n+1,k+1)$, where the same $\mathbb{K}^1$ is added to a k-dimensional subspaces and to the ambient n-dimensional space. This works when $n = \infty$ too, and for "oriented" real grassmannians.

G. Flag manifolds. One can generalize the construction of grassmannians by introducing a *flag manifold* whose points are r-tuples of nested subspaces (called *flags*) of increasing dimensions $0 < d_1 < d_2 < \cdots < d_r < n$:

$$0 \subset V^{d_1} \subset V^{d_2} \subset \cdots \subset V^{d_r} \subset \mathbb{R}^n.$$

Using the inner product structure in $\mathbb{R}^n$ one can decompose the ambient space of a flag into a direct orthogonal sum of subspaces and, by picking orthonormal bases in these subspaces, identify the flag manifold with $O_n/(O_{d_1} \times O_{d_2-d_1} \times \cdots \times O_{n-d_r})$. The same works for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ with the orthogonal groups replaced by unitary and symplectic groups respectively.

There is another quotient space description of the same flag manifold: as the quotient of $GL_n(\mathbb{K})$ (acting transitively on the flag manifold) by the stabilizer $P(\mathbb{K})$ of the standard coordinate flag $\mathbb{K}^{d_1} \subset \mathbb{K}^{d_2} \subset \dots \mathbb{K}^n$. The stabilizer consists of invertible upper block-triangular matrices with the blocks of sizes d_1, $d_2 - d_1$, etc. In the complex case this describes the flag manifolds (and grassmannians) as complex manifolds.

The manifold of *complete flags* $V^1 \subset V^2 \subset \cdots \subset V^{n-1} \subset \mathbb{K}^n$ will be denoted $F_n(\mathbb{K})$.

H. The Plücker embedding of $\mathbb{C}G(4,2)$. In a 2-dimensional subspace $V^2 \subset \mathbb{C}^4$, pick a basis. From the 4×2 matrix whose columns represent the vectors of this basis, one can form a 6-array of 2×2-determinants, not all equal 0 (since the matrix has rank 2):

$$\Delta := (\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}).$$

A basis change in V is described by the right multiplication of the 4×2-matrix by an invertible 2×2-matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. This causes the change $\Delta \mapsto \lambda\Delta$, where $\lambda = \alpha\delta - \beta\gamma \neq 0$. Thus, we have defined a map $\mathbb{C}G(4,2) \to \mathbb{C}P^5$ known as the *Plücker embedding.* The grassmannian has complex dimension 4 (check this!) and so the image of the Plücker embedding is a hypersurface given by one homogeneous equation in $\mathbb{C}P^5$. One way to find the equation is to apply Laplace's theorem (see e.g. [4]) about cofactor expansions of determinants with respect to several columns. Write two copies of our 4×2 matrix next to each other to form a 4×4-matrix with zero determinant. Applying Laplace's 2-column cofactor expansion, we obtain the *Plücker relation*

$$0 = 2\Delta_{12}\Delta_{34} - 2\Delta_{13}\Delta_{24} + 2\Delta_{14}\Delta_{23}.$$

Here is an invariant description of this construction in the language of exterior forms. To $V^2 \subset \mathbb{C}^4$, associate an exterior 2-form φ (unique up to a non-zero scalar factor) with V as the kernel (i.e. φ is the pull-back to $\mathbb{C}^4$ of a non-zero exterior 2-form on the quotient plane $\mathbb{C}^4/V$). We obtain an embedding $V \mapsto Span(\varphi)$ of the grassmannian into the projectivization of $\Lambda^2\mathbb{C}^{4*}$. Since φ is degenerate, we have $\varphi \wedge \varphi = 0$, which is the Plücker relation.

The same construction works *verbatim* in the real case and yields the Plücker embedding $G(4,2) \subset \mathbb{R}P^5$. However, if the basis change in $V^2 \subset \mathbb{R}^4$ is required to be orientation-preserving, i.e. $\lambda > 0$, then we obtain an embedding of $G_+(4,2)$ into the sphere S^5 of *rays* (rather than 1-dimensional subspaces) or, equivalently, the unit sphere in $\mathbb{R}^6$:

$$\Delta_{12}^2 + \Delta_{34}^2 + \Delta_{13}^2 + \Delta_{24}^2 + \Delta_{14}^2 + \Delta_{23}^2 = 1.$$

The coordinate change $u = (x+y)/\sqrt{2}$, $v = (x-y)/\sqrt{2}$ transforms $2uv$ to $x^2 - y^2$ and $u^2 + v^2$ to $x^2 + y^2$. Applying this lemma to the Plücker relation and the equation of the unit sphere, we obtain respectively

$$x_1^2 - y_1^2 - y_2^2 + x_2^2 + x_3^2 - y_3^2 = 0 \text{ and } x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 = 1.$$

Therefore, in our new coordinates $(\mathbf{x}, \mathbf{y})$ in $\mathbb{R}^6$, the "oriented" grassmannian is given by the equations $\|\mathbf{x}\|^2 = 1/2$, $\|\mathbf{y}\|^2 = 1/2$. Thus, $G_+(4,2) \simeq S^2 \times S^2$, while $G(4,2)$ is obtained by factorizing $S^2 \times S^2$ by the simultaneous antipodal involution $(\mathbf{x}.\mathbf{y}) \mapsto (-\mathbf{x}, -\mathbf{y})$.

EXERCISES

1. Prove that a compact subset in $\mathbb{R}^\infty$ is contained in a finite dimensional subspace.

2. Identify SO_4 with $(S^3 \times S^3)/(-1,-1)$ by examining the action $\mathbf{q} \mapsto \mathbf{u}\mathbf{q}\mathbf{v}^{-1}$ of $(\mathbf{u}, \mathbf{v}) \in Sp_1 \times Sp_1$ on $\mathbf{q} \in \mathbb{H}$.

3. Which of the following spaces are homeomorphic and which are not: (a) the space T_1S^2 of unit tangent vectors to S^2, (b) $V(3,2)$, (c) in $\mathbb{C}^3$ with coordinates z_1, z_2, z_3, the intersection of the complex surface $z_1^2+z_2^2+z_3^2 = 0$ with the unit sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, (d) $\mathbb{R}P^3$, (e) $S^2 \times S^1$?

4. Show that the real and imaginary parts of an Hermitian form are a Euclidean inner product and a non-degenerate anti-symmetric bilinear form in $\mathbb{C}^n$ considered as a $2n$-dimensional real space. Furthermore, show that the Hamiltonian inner product in $\mathbb{H}^n$ considered as a $2n$-dimensional complex vector space is the sum $Z(\mathbf{q}', \mathbf{q}) + W(\mathbf{q}', \mathbf{q})j$, where Z is an Hermitian form and W is a non-degenerate anti-symmetric complex-bilinear form. Deduce that the real (resp. complex) *symplectic group* $Sp(2n, \mathbb{R})$ (resp. $Sp(2n, \mathbb{C})$), defined as the group of linear symmetries of a non-degenerate anti-symmetric bilinear form in $\mathbb{R}^{2n}$ (resp. $\mathbb{C}^{2n}$) contains the compact group U_n (resp. Sp_n).

5. Identify $F_3(\mathbb{K})$ with the hypersurface in $\mathbb{K}P^2 \times \mathbb{K}P^2$ given by the equation $x_1y_1 + x_2y_2 + x_3y_3 = 0$ in homogeneous coordinates $\mathbf{x}$ and $\mathbf{y}$ on the two projective planes.

6. Show that by gluing the sides of the square (Figure 3) of matching colors and orientations, one obtains respectively: the cylinder, Möbius band, torus, Klein bottle, and projective plane.

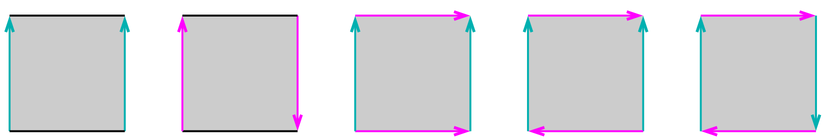

Figure 3: $C^2, M^2, T^2, K^2, \mathbb{R}P^2$

7. Show that $\mathbb{R}P^2$ is obtained by gluing a disk D^2 and a Möbius band M^2 along their boundaries, and the Klein bottle K^2 is thus obtained from two Möbius bands.

8. Attaching a *handle* to a surface is done but cutting out two holes and gluing in a cylinder by identifying its two boundaries with the boundaries of the holes. Show that there are two topologically different ways of attaching a handle to S^2, and one of them (orientation-respecting) yields T^2, and the other K^2.

9. Another surgery of a surface is done by attaching a Möbius band along the boundary of a hole. Show that attaching this way 3 Möbius bands is equivalent, up to homeomorphism of the surface, to attaching one Möbius band and one handle, and moreover, that after attaching the first one, the two ways of attaching the handle are equivalent. Derive that all surfaces obtained from S^2 by attaching handles and/or Möbius bands in any succession are homeomorphic to one of S^2_g, P^2_g, or K^2_g: the sphere, projective plane, or Klein bottle with g handles. (These are known to be, up to homeomorphism, the only closed surfaces, and they are pairwise non-homeomorphic.)

Lecture 2. Basic constructions

We describe certain operations which produce new topological spaces from those already defined.

A. Disjoint unions and products. The obvious topology of the disjoint union of two (or any collection) of topological spaces can be characterized by the property that a map $X \sqcup Y \to Z$ to any space Z is continuous if and only if its restrictions to X and Y are continuous. Equivalently, it is the weakest topology in which the inclusion maps $X, Y \subset X \sqcup Y$ are continuous.

In contrast, the Cartesian product is equipped with the strongest topology in which the projection maps $X \times Y \to X, Y$ are continuous. The products $U \times V$ of open sets $U \subset X$ and $V \subset Y$ form a *prebase* of this topology (i.e. all open sets are obtained from them by the operations of finite intersections and arbitrary unions). Equivalently, a map $Z \to X \times Y$ from any Z is a pair of maps $Z \to X$, $Z \to Y$, and their continuity is equivalent to the continuity of the former. In particular, a sequence (x_n, y_n) converges in $X \times Y$ if and only if the sequences x_n and y_n converge in X and Y respectively.

For example, the *cylinder* of X is defined as its product $X \times I$ with the unit *interval* $I := [0, 1]$ of the number line (Figure 4a).

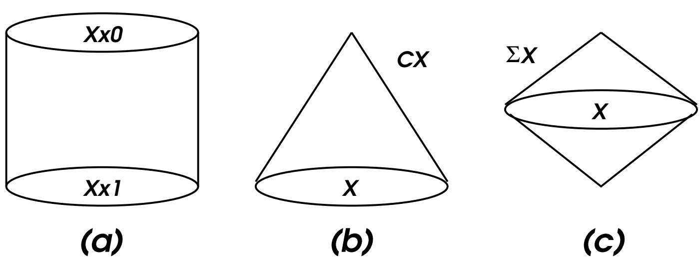

Figure 4: Cylinder, cone, suspension

B. Quotients: cones and suspensions. We have already mentioned the construction of the *quotient space* $X/\sim$ of a topological space by an equivalence relation. By definition, a subset U in $X/\sim$ is open if and only if its inverse image $\pi^{-1}(U)$ under the canonical projection $\pi : X \to X/\sim$ is open in X. In other words, this is the weakest topology in which π is continuous. Equivalently, a map $(X/\sim) \to Z$ is continuous if and only if its composition with π is. In particular, a continuous map $X \to Z$ constant on equivalence classes descends to a map $(X/\sim) \to Z$ which is continuous automatically.

Given a subset $A \subset X$, the quotient X/A is defined by declaring all points of A to form one equivalence class, and each point outside A to be an equivalence class of its own. For example, $D^n/\partial D^n \simeq S^n$.

The *cone* CX and *suspension* ΣX (Figure 4bc) are defined as such quotients $CX := X \times [0,1]/X \times 0$ and $\Sigma X := CX/X$ of the cylinder by one or both of its bases respectively.

The *mapping cylinder* $Cyl(f)$ of a map $f : X \to Y$ is defined as the quotient of $X \times [0,1] \sqcup Y$ by the equivalence relation $(x,1) \sim f(x)$.

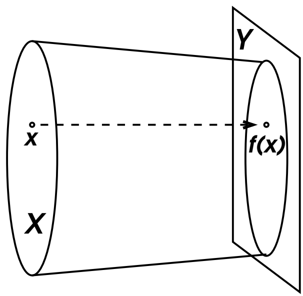

Figure 5: $Cyl(f)$

C. Joins. The *join* $X * Y$ of X and Y is obtained by connecting each point of X with each point of Y by an interval. It is formally described as the quotient of $X \times I \times Y$ by the identifications $(x,0,y) \sim (x,0,y')$ and $(x,1,y) \sim (x',1,y)$ for all $x, x' \in X$ and all $y, y' \in Y$. For example, a tetrahedron is the join $I * I$ of any pair of its opposing edges.

One equips the join with *Milnor's topology*, which is the strongest topology in which the following three mappings are continuous: (i) $X * Y \to [0,1]$ induces by the projection $X \times I \times Y \to I$, (ii) $X * Y - Y \to X$ induced by the projection $X \times [0,1) \times Y \to X$, and (iii) $X * Y - X \to Y$ induces by the projection $X \times (0,1] \times Y \to Y$. In some pathological cases Milnor's topology differs from the one defined by the consecutive product and quotient constructions, but has the advantage of making the join construction associative.

In particular, the join $X_1 * \cdots * X_n$ can be considered as the subset in $CX_1 \times \cdots \times CX_n$ consisting of the collections $(x_1,t_1), \ldots, (x_n,t_n)$ (here $x_i \in X_i$, $t_i \in [0,1]$, and $t_i = 0$ represents the vertex of the i-th cone regardless of x_i), satisfying $t_1 + \cdots + t_n = 1$. It is quipped with Milnor's topology, in which each function t_i is continuous, as is the projection of $t_i^{-1}(0,1]$ to $X_1 * \cdots \widehat{X_i} \cdots * X_n$ (where the "hat" indicates that X_i is omitted).

D. Mapping spaces. The set $C(X,Y)$ of continuous maps from X to Y is equipped with the *compact-open topology*. By definition, a prebase of it is formed by subsets

$$\mathcal{O}_{K,U} := \{f \in C(X,Y) \mid f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset Y$ is open. Thus, every neighborhood $\mathcal{U}$ of $f \in C(X,Y)$ contains a possibly smaller neighborhood consisting of all function from X to Y which send certain compact subsets $K_1, \ldots, K_n \subset X$ to certain open subsets $U_1, \ldots, U_n \subset Y$ respectively. When Y is a metric space, the compact-open topology is the topology of uniform convergence of maps on compact subsets.

By analogy with the set-theoretic notation X^Y for the set of *all* functions $Y \to X$, one sometimes uses the same exponential notation for $C(Y,X)$. However, the *exponential law* $(X^Y)^Z = X^{Y\times Z}$, i.e. $C(Z, C(Y,X)) = C(Z \times Y, X)$ can fail for general topological spaces. Nevertheless it is known to hold when Y is locally compact (see [3]). That is, given $F : Z \times Y \to X$, the family $Z \ni z \mapsto F|_{z\times Y} \to X$ of its restrictions is continuous as a map from Z to $C(Y,X)$, and this correspondence is bijective and homeomorphic. This will be sufficient for our applications.

In particular, a continuous map $F : Z \times I \to X$ is the same as a continuous map from Z to the *path space* $E(X) := C(I,X)$.

E. The base point case. We will regularly consider topological *pairs* (X,A), where $A \subset X$, and assume that a map $f : (X,A) \to (Y,B)$ between pairs maps A to B. But most of the time we will deal with the category of *base point spaces*, i.e. pairs (X, x^0) where x^0 is a fixed *base point*. Some of the previous constructions have to be modified to reflect the presence of base points.

The analogue of the disjoint union is the *bouquet* of spaces:

$$\bigvee_\alpha (X_\alpha, x^0_\alpha) := \bigsqcup_\alpha X_\alpha \,/\, \bigsqcup_\alpha \{x^0_\alpha\},$$

obtained by "gluing" (a family of) based point spaces (X_α, x^0_α) by their base points, declared to be the base point of the bouquet.

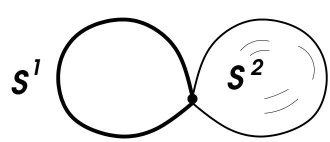

Figure 6: $S^1 \vee S^2$

By definition, the product of base point spaces $\prod_\alpha (X_\alpha, x_\alpha^0)$ consists of collections of points $x_\alpha \in X_\alpha$ of which all but finitely many coincide with x_α^0. It is equipped with the topology defined by the condition that a subset is open (closed) whenever its intersection with every finite sub-product is open (resp. closed). For example, $(\mathbb{R}^\infty, \mathbf{0}) = \prod_{\alpha=1}^\infty (\mathbb{R}, 0)_\alpha$.

The *smash-product* $X\#Y$ of base point spaces is defined as the quotient $X \times Y/X \vee Y$ of the Cartesian product by the "coordinate cross" $(X \times y^0) \cup (x^0 \times Y)$.

In general, given a pair (X, A), the quotient X/A is considered as a base point space with the class $[A]$ taken for the base point. This defines a functor from the category of pairs to the category of base point spaces. When $A = \emptyset$, the quotient $X/\emptyset$ is defined as the disjoint union $X^+ := X \sqcup pt$ of X with a one-point space which is declared to be the base point of X^+.

The cone and suspension of a base point space (X, x^0) are defined by the additional factorization of the cylinder $[0,1] \times X$ by the generator $[0,1] \times x^0$ over the base point. We still denote them CX and ΣX rather than $C(X, x^0)$ and $\Sigma(X, x^0)$, hoping that it is clear from the context whether we are in a base point space category or not.

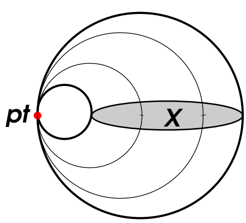

Figure 7: $\Sigma(X^+)$

By definition, the *path space* $E(X, x^0)$ consist of paths $\gamma : [0,1] \to X$ starting at the base point: $\gamma(0) = x^0$, and the constant path plays the role of the base point. The fibers (i.e. level sets) of the projection $E(X, x^0) \to X$ are the spaces $E(x^0, x)$ of paths connecting x^0 with a given point x. The fiber $E(x^0, x^0)$ over the base point is called the *loop space* and denoted ΩX.

It follows from the exponential law that in the base point category the suspension and the loop space constructions are adjoint to each other in the following sense:

$$C(\Sigma X, Y) = C(X, \Omega Y).$$

EXERCISES

10. On a space X, introduce the following equivalence relation: $x_0 \sim x_1$ if there is a continuous path $\gamma : [0,1] \to X$ connecting x_0 with x_1: $\gamma(0) = x_0$, $\gamma(1) = x_1$. The equivalence classes are called *path-connected components* of X . Show that path-connected components are connected in the usual sense, and use the function $\sin 1/x$ to show that the converse can be false.

11. Show that $CS^n = D^{n+1}$, $\Sigma S^n = S^{n+1}$, $pt * X = CX$, $S^0 * X = \Sigma X$, $S^n * X = \Sigma^{n+1} X := \Sigma \dots \Sigma X$ (iterated suspension), $S^m * S^n = S^{m+n+1}$, $\mathbb{R}P^m * \mathbb{R}P^n = \mathbb{R}P^{m+n+1}$.

12. For every partition of $[0,1]$ into N equal intervals $I_k = [(k-1)/N, k/N]$ and every collection $U_1, \dots, U_N$ of open sets in X, consider in the path space $E(X) = C([0,1], X)$ the subset $\bigcap_{k=1}^N \mathcal{O}_{I_k, U_k}$ (consisting of all paths $\gamma : [0,1] \to X$ mapping each I_k to U_k). Prove that such subsets form a base of compact-open topology of the path space.

13. Show that smash-product is associative.

14. For base point spaces, show that $S^n \# X = \Sigma^n X$, $S^m \# S^n = S^{m+n}$.

15. Show that $X^+ \# Y^+ = (X \times Y)^+$.

16. Prove that all spaces $E(S^n, x_0, x_1)$ of path connecting any two points in S^n are homeomorphic.

Lecture 7. The universal covering

A covering (like those at the end of Lecture 6) with a simply connected total space is called *universal* because all coverings of its base can be constructed from it.

A. Classification of coverings. An *equivalence* of two coverings of the same base is defined as a homeomorphism between their total spaces commuting with the projections:

$$\begin{tikzcd}[column sep=small] (Y', y') \arrow[rr, "f", "\simeq"'] \arrow[dr, "p'"'] & & (Y'', y'') \arrow[dl, "p''"] \\ & (X, x^0) & \end{tikzcd}$$

Obviously, this is an equivalence relation, and if the based points are fixed in advance (as in the diagram), an equivalence f satisfying $f(y') = y''$ is unique when it exists. Otherwise it is unique up to deck transformations. Furthermore, the subgroups $p'_*(\pi_1(Y', y'))$ and $p''_*(\pi_1(Y'', y''))$ coincide (since $p''_* f_* = p'_*$), while a change of the base point y' over x^0 (and respectively of $y'' := f(y')$) results in a conjugated subgroup. The converse is true at least when X is locally path-connected:

`Proposition.` *Two coverings $p' : Y \to X$ and $p'' : Y'' \to X$ of the same locally path-connected space X are equivalent if and only if the corresponding subgroups in $\pi_1(X, x^0)$ are conjugated.*

`Proof.` The two subgroups in question are $p'_*(\pi_1(Y', y'))$ and $p''_*(\pi_1(Y'', y''))$, defined by a choice of the base points such that $p'(y') = x^0 = p''(y'')$. If the subgroups are conjugated by (the homotopy class of) a loop α, path-lifting the loop to Y' starting from y' results in a path $\widetilde{\alpha}$ ending at a new base point, $\widetilde{y}'$, such that $p'_*(\pi_1(Y, \widetilde{y}'))$ coincides with $p''_*(\pi_1(Y'', y''))$.

When X is locally path-connected, so are Y' and Y''. The Map-Lifting Theorem, applied to covering p'' with $(Z, z^0) = (Y', \widetilde{y}')$ and $F = p'$, implies the existence of a unique $f : (Y', \widetilde{y}') \to (Y'', y'')$ such that $p'' \circ f = p'$. Reversing the roles of p' and p'' we obtain $g : (Y'', y'') \to (Y', \widetilde{y}')$ such that $p' \circ g = p''$. The composition $g \circ f$ gives a unique lift of $F = p'$ to the covering p'. But another such lift is given by the identity map $\mathrm{id}_{Y'}$, and hence $g \circ f = \mathrm{id}_Y$. For symmetric reasons, $f \circ g = \mathrm{id}_{Y''}$, and therefore f is a homeomorphism. $\square$

`Corollary.` *The universal covering of a locally path-connected space, when exists, is unique up to equivalence.*

The significance of the universal covering $\widetilde{X} \to X$ is that it is a regular covering, i.e. $X = \widetilde{X}/G$ is the quotient by a properly discontinuous right action of a discrete group, where $G \cong \pi_1(X)$. For any subgroup $H \subset G$, the quotient map $\widetilde{X} \to \widetilde{X}/H$ factors through $Y := \widetilde{X}/H$, and thus defines a covering $p : Y \to X$ corresponding to a prescribed subgroup in $\pi_1(X)$:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dd] \\ Y = \widetilde{X}/H \arrow[dr, "p"'] & \\ & X = \widetilde{X}/G \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(\widetilde{X}) = 1 \arrow[dl] \arrow[dd] \\ \pi_1(Y) \cong H \arrow[dr, "p_*"'] & \\ & \pi_1(X) \cong G \end{tikzcd}$$

More precisely, the orbit $\widetilde{x}^0 G$ of a base point $\widetilde{x}^0 \in \widetilde{X}$ is the corresponding base point x^0 in X, and taking $y^0 := \widetilde{x}^0 H$ for the base point in Y, we obtain $p_*(\pi_1(Y, y^0)) = H$. It only remains to find out whether the universal covering exists.

B. Constructing universal coverings. Let us call X *semilocally simply connected* if every $x \in X$ has a neighborhood U_x such that every loop in U_x is contractible in X. This condition is necessary for existence of a universal covering $\widetilde{X} \to X$, because X is locally homeomorphic to $\widetilde{X}$ which is simply connected.

`Theorem.` *A (path-connected) locally path-connected semilocally simply connected space has a universal covering.*

`Proof.` It is based on the following explicit construction. Consider the space $E(X, x^0)$ of path based at x_0, and call two path $\gamma, \gamma' : (I, 0) \to (X, x^0)$ equivalent if $\gamma(1) = \gamma'(1)$ and γ is homotopic to γ' relative to the endpoints: $\gamma \sim_{\partial I} \gamma'$. Then the quotient space $\widetilde{X} := E(X, x^0)/\sim_{\partial I}$ by this equivalence relation is a universal covering space of X under the projection $p : \widetilde{X} \to X$ defined by the evaluation $\pi : \gamma \mapsto \gamma(1)$ of paths at the endpoint.

Assume for the moment that p is a covering indeed. The fiber $p^{-1}(x_0)$ consists of homotopy classes of loops in (X, x^0). Moreover, a loop α has the tautological lift $\alpha_t(u) := \alpha(tu)$ to a path $t \mapsto \alpha_t$ in $E(X, x^0)$ (and hence in $\widetilde{X}$) which at $t = 1$ turns into the point $[\alpha] \in p^{-1}(x^0)$. By Corollary 2 of the Map-Lifting Lemma, this bijection between $\pi_1(X, x^0)$ and $p^{-1}(x^0)$ shows that the subgroup $p_*(\pi_1(\widetilde{X}))$ is trivial, and hence (by Corollary 1), that $\widetilde{X}$ is simply connected.

To show that p is a covering, we construct a chart of $x \in X$ by picking a neighborhood U_x such that every loop in it is contractible in X, and inside U_x find a neighborhood V_x such that every $x' \in V_x$ can be connected with x by a path (call it $\delta_{x'}$) inside U_x.

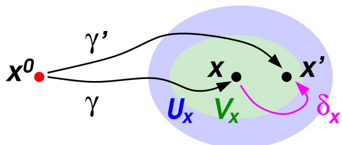

Figure 23: $pr : \pi^{-1}(V_x) \to p^{-1}(x)$

Given two path γ and γ' connecting x^0 with x and x' respectively, we can close them by $\delta_{x'}$ into a loop, whose homotopy class does not depend on the choice of $\delta_{x'}$ because of the properties of U_x. Thus, we get a surjective map $pr : \pi^{-1}(V_x) \to p^{-1}(x)$: $pr(\gamma') = pr(\gamma)$ if and only if the loop is contractible. Together with the evaluation map $\pi : \pi^{-1}(V_x) \to V_x$, they define a bijection

$$p \times \widehat{pr} : p^{-1}(V_x) \to V_x \times p^{-1}(x)$$

— two paths ending in V_x are equivalent if and only if their endpoints coincide and the loop they form is contractible. We prove below that this bijection is a homeomorphism. However, one can ignore the quotient topology of $\widetilde{X}$, and simply take $\widehat{pr}^{-1}[\gamma] \simeq V_x$ for a neighborhood of $[\gamma] \in p^{-1}(x)$. By varying V_x and $[\gamma]$ one obtains the base of a topology on $\widetilde{X}$ which makes it a universal covering of X. $\square$

C. The quotient topology of $\widetilde{X}$. To prove that $p \times \widehat{pr}$ is a homeomorphism, we need to check: (a) that $\widehat{pr} : p^{-1}(V_x) \to p^{-1}(x)$ is continuous, i.e. that for a homotopy class $[\gamma] \in p^{-1}(x)$ its inverse image $\pi^{-1}[\gamma]$ is open in $E(X, x^0)$, and (b) that the bijection $p : \widehat{pr}^{-1}[\gamma] \to V_x$ is a homeomorphism, i.e. that the inverse bijection p^{-1} is continuous.

To establish (a) it suffices to show that γ has a compact-open neighborhood consisting entirely of paths γ' with $pr(\gamma') = pr(\gamma)$. In other words, we need to show that $\mathcal{U} := \pi^{-1}(U_x)$ is *semilocally path-connected*: Every point of $\mathcal{U}$ has a neighborhood whose points lie in the same path-connected component of $\mathcal{U}$.

Assume for the moment that (a) is true, and hence $\widehat{pr}^{-1}[\gamma]$ is open in the quotient topology of $\widetilde{X}$. The continuity of the bijection $p^{-1} : V_x \to \widehat{pr}^{-1}[\gamma]$ is equivalent to p being *open*, meaning that images of open sets are open. But open sets in $\widetilde{X}$ are exactly those whose inverse images in $E(X, x^0)$ are open, and so it suffices to prove that the evaluation map $\pi : E(X, x^0) \to X$ is open.

Thus, the following lemma shows that $\widetilde{X}$ equipped with the quotient topology is the universal covering space of X.

`Lemma.` *If X is locally path-connected, then $\pi : E(X, x^0) \to X$ is open. If in addition X is semilocally simply connected, then $\pi^{-1}(U)$ are semilocally path-connected for all open $U \subset X$.*

`Proof.` Recall that a base of the compact-open topology on $E(X, x^0)$ is formed by sets $\mathcal{O}$ which consist of all paths $\gamma : [0,1] \to X$ (with $\gamma(0) = x^0$) mapping a specified collection of compact subsets of $[0,1]$ to specified open subsets of X respectively. Let γ be one such path in $\mathcal{O}$, and let U_i be such open subsets for those compact subsets of the collection which contain 1. Then $x := \gamma(1) \in U_x := \bigcap_i U_i$. Let V_x be a neighborhood of x inside U_x whose existence is guaranteed by the local path-connectedness of X: Every $x' \in V_x$ can be connected to x by a path $\delta_{x'}$ lying in U_x. By continuity of γ, some interval $[1-\epsilon, 1]$ is mapped by γ to V_x. Then the path γ', which on $[0, 1-\epsilon]$ coincides with γ and on $[1-\epsilon, 1]$ is stretched further to x' along $\delta_{x'}$, lies in $\mathcal{O}$. Thus, together with $x \in \pi(\mathcal{O})$, its neighborhood V_x also lies in $\pi(\mathcal{O})$. This proves that the map π is open.

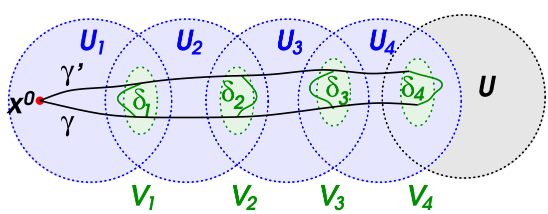

Figure 24: Proof of the 2nd part, $N = 4$

To prove the second statement, consider a path γ with $x := \gamma(1) \in U$. Since every point $\gamma(u)$ has a neighborhood guaranteed by the semilocal simply connectedness condition, we can pick N large enough so that $\gamma\left([\frac{i-1}{N}, \frac{i}{N}]\right)$ lie in such neighborhoods U_i (Figure 24). Inside $U_i \cap U_{i+1}$ (where $U_{N+1} := U$ for $i = N$), pick a neighborhood V_i of $\gamma(\frac{i}{N})$ guaranteed by the local path-connectedness condition. Now define the compact-open neighborhood $\mathcal{V}$ of γ in $E(X, x^0)$ consisting of all path γ' which map each $[\frac{i-1}{N}, \frac{i}{N}]$ to U_i and each $\frac{i}{N}$ to V_i. Given such a γ', connect $\gamma'(\frac{i}{N})$ with $\gamma(\frac{i}{N})$ by a path δ_i inside $U_i \cap U_{i+1}$. The loop $\delta_{i-1}^{-1}\gamma_i'\delta_i\gamma_i^{-1}$, where γ_i and γ_i' are the restrictions of γ and γ'

to $[\frac{i-1}{N}, \frac{i}{N}]$ (and δ_0 is the constant path at x_0), lies in U_i. Therefore the loop is contractible, i.e. extends to a map to X of the rectangle $[\frac{i-1}{N}, \frac{i}{N}] \times I$. Altogether these maps assemble into a homotopy $(\gamma_t) : [0,1] \times I \to X$ between $\gamma = \gamma_0$ and $\gamma' = \gamma_1$, with all $\gamma_t(0) = x^0$ and all $\gamma_t(1) \in U$. This is a path connecting γ with γ' in $\pi^{-1}(U)$.

E. Relations with Galois theory. We have found that under some mild technical assumptions on X (satisfied for locally contractible spaces, which includes manifolds as well as CW-complexes, see Appendix 1 in [8]), equivalence classes of coverings $p : Y \to X$ correspond to conjugacy classes of subgroups in $G := \pi_1(X)$, and a regular covering corresponds to a normal subgroup $H \subset G$, in which case the quotient group G/H becomes the automorphism group of the covering. Moreover, coverings spaces of (X, x^0) equipped with based points over x^0 correspond to subgroups $p_*(\pi_1(Y, y^0)) \subset \pi_1(X, x^0)$, and the inclusion $\widetilde{p}_*(\pi_1(\widetilde{Y}, \widetilde{y}^0)) \subset p_*(\pi_1(Y, y^0))$ of the subgroups is equivalent to the existence of a unique covering map $q : (\widetilde{Y}, \widetilde{y}^0) \to (Y, y^0)$ such that $\widetilde{p} = p \circ q$. This picture is analogous to Galois theory of fields extensions in algebra (see e.g. [5] for an elementary exposition), and the following example establishes a direct connection.

In the space $\mathbb{C}^{n+1}$ with coordinates $(x, a_1, \dots, a_n)$, consider the hypersurface P_n (Figure 25a) defined by the equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

The projection $P_n \to \mathbb{C}^n : (x, a) \mapsto a$ is an n-fold covering of $B_n := \mathbb{C}^n - \Delta_n$, where Δ_n is the *discriminant* hypersurface Δ (over which, by its definition, the corresponding polynomials have multiple roots).

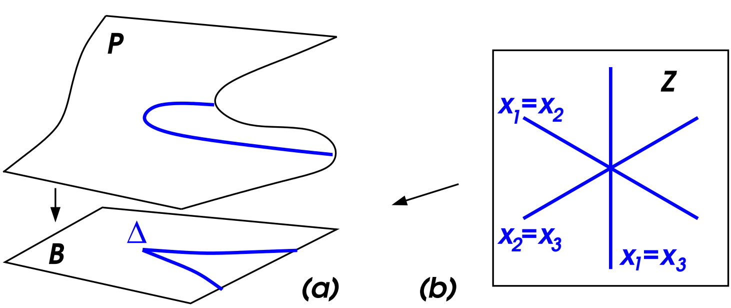

Figure 25: $x^3 + a_2 x + a_3 = 0$

Another, $n!$-fold covering is given by the *Vieta map* $\mathbb{C}^n \to \mathbb{C}^n$: $(x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)$ obtained by expressing the coefficients of the polynomial $(x - x_1) \ldots (x - x_n)$ as elementary symmetric functions of its roots: $a_k = (-1)^k \sigma_k(x_1, \ldots, x_n)$. Namely, B_n is the quotient of the *configuration space* $Z_n := \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j\}$ (see Figure 25b, where $x_1 + x_2 + x_3 = 0$) of ordered n-tuples of distinct complex numbers by the group S_n permuting the numbers. So, this covering is regular.

Rational function on the base form a field $\mathbb{C}(a_1, \ldots, a_n)$ from which the field $\mathbb{C}(x_1, \ldots, x_n)$ of is obtained as the "splitting field" of the polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$ by adjoining all its roots. This is a *normal extension* with the Galois group S_n. The field $\mathbb{C}(x, a_1, \ldots, a_n)$ of rational functions on P_n can be embedded into the splitting field in n *conjugated* ways by putting $x = x_k$. All elements of this subfield are fixed by the subgroup $S_{n-1} \subset S_n$ permuting all x_i except x_k.

Likewise, there are n ways to factor the covering $Z_n \to B_n$ through $P_n \to \mathbb{C}^n$. It remains only to describe the fundamental group $\pi_1(B_n)$ and its normal subgroup $\pi_1(Z_n)$ with the quotient group S_n.

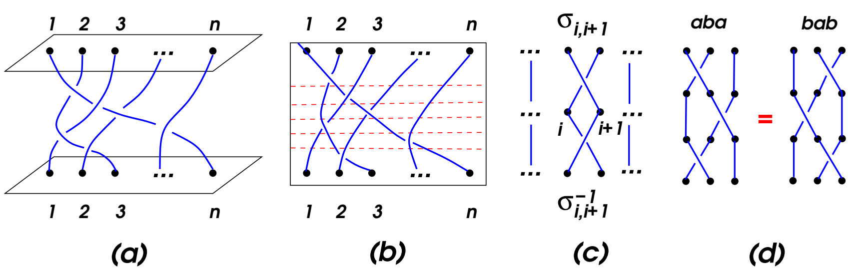

Figure 26: Braids

F. The braid group on n strands. Points of B_n can be interpreted as *unordered* configurations of n distinct complex numbers, and we can take the roots of $(x-1)(x-2)\ldots(x-n)$ for the base point. Then a based loop in B_n is a family of such configurations starting and ending at the base point (Figure 26a). Since it matters only up to homotopy, one can flatten the 3D-figure into a 2D *braid* (Figure 26b), where it only matters which strand goes *over* and which one *under* when two of them cross on the way from top to bottom. Composition of loops translates into the vertical concatenation of braids. Perturbing the strands, one may assume that different crossings occur at different "heights", and cut the braid (as shown on Figure 26b in red) into the composition of elementary braids σ_i or σ_i^{-1} (Figure 26c) with one crossing of strands i and $i+1$ only. Thus, the *braid group* $Br_n := \pi_1(B_n)$ is generated by $\sigma_{i,i+1}$, $i = 1, \dots, n-1$.

While $\sigma_{i,i+1}\sigma_{j,j+1} = \sigma_{j,j+1}\sigma_{i,i+1}$ when $|i-j| > 1$, Figure 6d shows that $\sigma_{i-1,i}\sigma_{i,i+1}\sigma_{i-1,i} = \sigma_{i,i+1}\sigma_{i-1,i}\sigma_{i,i+1}$ for all $i = 2, \dots, n-1$. This is known to be a complete presentation of Br_n.

Forgetting the way the strands braid but remembering how the nodes $1, ..., n$ are permuted, we obtain the homomorphism $Br_n \to S_n$ corresponding to the regular covering $Z_n \to B_n$: The kernel of it, known as the group of *colored braids*, is $\pi_1(Z_n)$. Under this homomorphism, the elementary braids $\sigma_{i,i+1}^{\pm 1}$ become the transpositions $\tau_{i,i+1}$. They satisfy $\tau_{i,i+1}^2 = \mathrm{id}$, which together with the same relations as those obeyed by $\sigma_{i,i+1}$ provide the standard presentation of S_n on $n-1$ generators.

EXERCISES

49. Classify all coverings of S^1.

50. Show that every covering space of a torus T^n is homeomorphic to one of the spaces $T^k \times \mathbb{R}^{n-k}$ where $0 \le k \le n$.

51. For each $n > 1$, find a space X with $\pi_1(X) \cong \mathbb{Z}_n$ (the cyclic group of order n).

52. Represent the Klein bottle K^2 as the quotient of $\mathbb{R}^2$ by a discrete subgroup, compute $\pi_1(K^2)$, and classify all coverings of K^2.

53. Show that the universal covering spaces of all smooth connected surfaces (compact or not) except S^2 and $\mathbb{R}P^2$ are homeomorphic to $\mathbb{R}^2$. (`Hint:` One way is to use Riemann's mapping theorem from complex analysis.)

54. Let M be a connected non-orientable manifold, and $p : M^{or} \to M$ its orienting covering. Describe the homomorphism $\pi_1(M) \to \mathbb{Z}_2$ whose kernel is $p_*(\pi_1(M^{or}))$.

55. Let $p : \widetilde{G} \to G$ be the universal covering of a connected Lie group G. Show that $\widetilde{G}$ has a unique Lie group structure such that p is a smooth homomorphism, and that the kernel $\ker p \cong \pi_1(G)$ is a discrete central subgroup in $\widetilde{G}$.

56. Show that $\mathbb{C} - \{1/n \mid n = 1, 2, \dots\}$ is not semilocally simply connected.

57. For a path-connected locally path-connected semilocally simply connected X, show that the covering space (Y, y^0) corresponding to a given subgroup $H \subset \pi_1(X, x^0)$ can be constructed by taking the quotient of $E(X, x^0)$ by the equivalence relation: $\gamma \sim_H \gamma'$ if and only if $\gamma(1) = \gamma'(1)$ and the homotopy class of the loop $\gamma'\gamma^{-1}$ is in H.

58. Compute $\pi_1(\mathbb{C} - \{1, \dots, n\})$.

59. Show that the complement to the discriminant in the space of polynomials $x^n + a_1x^{n-1} + \cdots + a_n$ is homotopy equivalent to its intersection with the hyperplane $a_1 = 0$ (and even diffeomorphic to the Cartesian product of this intersection with $\mathbb{C}$).

60. Show that turning the diagram of a given braid up-side-down yields the inverse braid.

61. Imagine that the strands of the identity braid are drawn as n parallel segments on a rectangular strip of paper. Twisting one end of the strip $180°$ (so that the induced permutation is $n, n-1, \ldots, 1$) we obtain the *fundamental braid*. Represent it as the product of the generators $\sigma_{i,i+1}$. Show that its square is a colored braid lying in the center of Br_n.
`Remark:` The square is known to generate the center.

Lecture 14. Classification of G-bundles

We apply homotopy theory to a problem of general mathematical interest: classification of principal G-bundles and their associates.

A. Principal G-bundles. Let G be a *topological group*, i.e. a topological space and a group such that the multiplication and inversion maps, $G \times G \to G$ and $G \to G$, are continuous. Given a free continuous (right) action $E \times G \to E$ of G on a space E, the canonical projection $p : E \to B := E/G$ to the orbit space is called a *principal G-bundle*, provided that it is locally trivial. (The last condition is satisfied automatically at least in the case of smooth actions of compact Lie groups.)

In more detail: First, $pr_1 : B \times G \to B$ is the *trivial* G-bundle (with respect to the action of G by right translations on itself). It has a canonical section $B \to B \times G : x \mapsto e \in G$. *Vice versa*, a section $s : B \to E$ of a principal G-bundle $p : E \to B$ defines a *trivialization*: a G-equivariant bijection $B \times G \to E : (x, g) \mapsto s(x)g$. Continuity of the inverse map is a local property, and is easily checked using local trivializations (the red arrow is a G-equivariant homeomorphism):

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, leftrightarrow, "\simeq"] \arrow[dr, "p"'] & & U \times G \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}$$

Indeed, over U, the section s is a function $U \ni x \mapsto s(x) \in G$, and the inverse to the bijection $(x, g) \mapsto (x, s(x)g)$ is the left multiplication by $x \mapsto s^{-1}(x)$, which is continuous.

Thus, a principal G-bundle can be described via some open cover $B = \bigcup_\alpha U_\alpha$ as glued from trivial bundles $U_\alpha \times G \to U_\alpha$ by means of re-trivializations $(x, g) \mapsto (x, \varphi_{\alpha\beta}(x)g)$ over pairwise intersections. Here $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are *clutching functions*, which must satisfy $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$ and $\varphi_{\alpha\beta}(x)\varphi_{\beta\gamma}(x)\varphi_{\gamma\alpha}(x) = e$ when $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Regular coverings are examples of principal G-bundles, where G is discrete (and they are non-trivial, because the total space of a covering is required to be path-connected). *Stiefel fibrations* $V(n, k) \stackrel{O_k}{\to} G(n, k)$ and their oriented, complex, and quaternionic partners are examples with compact Lie groups in the role of G. For a "real-life" example, consider the tangent bundle $TM^n \to M^n$ of a smooth manifold. It is *associated* with a principal $GL_n(\mathbb{R})$-bundle over M whose fiber over $x \in M$ consists of all bases in $T_xM \cong \mathbb{R}^n$. Namely, the transition matrices between the bases form the group of invertible

$n \times n$-matrices acting freely and transitively on the set of bases. If $f_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ form an atlas on M, the corresponding clutching functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ are defined by the Jacobi matrices $\varphi_{\alpha\beta}(x) = \partial(f_\beta \circ f_\alpha^{-1})/\partial y|_{y=f_\alpha(x)}$.

Associated bundles. Given a principal G-bundle $p : E \to B$ and a G-space F (i.e. a space carrying a continuous left G-action), one can "replace" each fiber G with F to obtain the *associated bundle with the structure group G and fiber F*. Namely, its total space is the quotient $F_G := (E \times F)/G$ with respect to the diagonal left action $(x, f) \mapsto (xg^{-1}, gf)$, and the projection to $B = E/G$ is given by $(x, f) \to p(x)$ (which obviously factors through F_G).

For example, to a principle $GL_n(\mathbb{R})$-bundle one can associate a vector bundle using the standard vector representation of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$. Conversely, suppose we are given a real n-dimensional *vector bundle* over B, i.e. a family $\pi : T \to B$ of vector spaces $\pi^{-1}(x) \cong \mathbb{R}^n$, equipped with local trivializations $g_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ which are fiberwise linear. Their compositions $\varphi_{\alpha\beta} = g_\beta \circ g_\alpha^{-1}$ can be viewed as clutching functions $U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ recovering the underlying principal $GL_n(\mathbb{R})$-bundle with the fiber over $x \in B$ consisting of all bases in $\pi^{-1}(x)$.

Using other representations of $GL_n(\mathbb{R})$ (say, in exterior powers $\Lambda^k\mathbb{R}^n$ of $\mathbb{R}^n$), one obtains new vector bundles with the same structure group $GL_n(\mathbb{R})$. For instance, differential k-forms on a manifold M are smooth sections of the bundle $\Lambda^k T^* M$. Note that for $k = n$ (or 0) it is a *line bundle* (trivial in the latter case). Nevertheless our terminological convention requires that an associated bundle "remembers" the principal G-bundle it comes from: *the clutching functions defining a bundle with the fiber F and structure group G take values in G*, and not in the group $Homeo(F)$ of homeomorphisms of F (as would be minimally required), even when the homomorphism $G \to Homeo(F)$ defining the action of G on F is not injective.

Of course, one can forget the $GL_n(\mathbb{R})$-structure of the bundle $\Lambda^k T^* M$ and consider it simply as a vector bundle of dimension $N = \binom{n}{k}$ with the structure groups $GL_N(\mathbb{R})$. This illustrates the general principle: If the action of G on F is defined via an action of G' and a continuous group homomorphism $\rho : G \to G'$ (e.g. an inclusion), then the structure group G of any associated F-bundle can be replaced with G' — by considering G-valued clutching function as G'-valued. In this sense, *expansion of the structure group is always possible.*

Another example: Given two vector bundles ξ and η (over the same base B) with the fibers $\mathbb{R}^m$ and $\mathbb{R}^n$, one defines their *direct sum*

$\xi \oplus \eta$ (also known as *Whitney sum*) and their *tensor product* $\xi \otimes \eta$ by performing these operations fiberwise on $\xi^{-1}(x)$ and $\eta^{-1}(x)$ for each $x \in B$. Considering $\xi \oplus \eta$ and $\xi \otimes \eta$ simply as vector bundles of dimensions $m + n$ and mn means expanding their native structure group $GL_m(\mathbb{R}) \times GL_n(\mathbb{R})$ by embedding it into $GL_{m+n}(\mathbb{R})$ and $GL_{mn}(\mathbb{R})$ respectively.

On the other hand, narrowing a structure group G to a subgroup H means imposing an additional structure and is not always possible. For example, narrowing the group to the trivial subgroup $\{e\} \subset G$ is equivalent to trivializing the bundle. Narrowing the structure group $GL_n(\mathbb{R})$ of a vector bundle to the group H of invertible upper-triangular matrices is equivalent to picking in each fiber of the vector bundle a complete flag in a continuous fashion, or equivalently, to furnishing a section of the associated bundle with the fiber $F_n(\mathbb{R}) = GL_n(\mathbb{R})/H$ (defined by the natural action of $GL_n(\mathbb{R})$ on $F_n(\mathbb{R})$). Yet, the structure group $G := GL_n(\mathbb{R})$ (resp. $GL_n(\mathbb{C})$ or $GL_n(\mathbb{H})$) can be narrowed to its *maximal compact subgroup* $H := O_n$ (resp. U_n and Sp_n) provided that the base is cellular. In other words, a real (resp. complex or quaternionic) vector bundle over a cellular base can be endowed with a (fiberwise, continuous) Euclidean (resp. Hermitian and Hamiltonian) structure, or equivalently, the associated G/H bundle has a section. (For tangent bundles, this means that any manifold can be endowed with a Riemannian metric.) This is easily proved by cell induction (using Lemma *à la Feldbau's* below, and over manifolds — using partitions of unity), but the reason is that the space $G/H = S^2_+(\mathbb{R}^n)$ of positive definite quadratic forms in $\mathbb{R}^n$ is convex and hence contractible.

Another way of looking at this is to invoke the Gram–Schmidt orthogonalization to show that the embedding $H \hookrightarrow G$ is a homotopy equivalence. As we will see, whenever the inclusion of a subgroup H into a group G is a WHE, the structures of G- and H-bundles over cellular bases are equivalent.

C. Classification. Two principal G-bundles $p : E \to B$ and $\widetilde{p} : \widetilde{E} \to B$ (over the same base) are called *equivalent* if there exists a G-equivariant homeomorphism $h : E \to \widetilde{E}$ (where "equivariant" means $h(xg) = h(x)g$ for all $x \in E, g \in G$) such that $p = p' \circ h$:

$$\begin{tikzcd} E \arrow[rr, "\simeq", "h"'] \arrow[dr, "p"'] & & \widetilde{E} \arrow[dl, "\widetilde{p}"] \\ & B & \end{tikzcd}$$

We denote by $St(B, G)$ (after Steenrod) the set of equivalence classes of principal G-bundles over B. Note that $St(B,G)$ also classifies bundles over B with the structure group G and any G-space F in the role of the fiber, because in our terminology, such bundles are defined by collections of clutching functions with values in G.

`Classification theorem.` *(i) For every topological group G, there is a principal G-bundle $p : EG \to BG$ (called* universal*) such that $St(B,G) = \pi(B,BG)$ for all cellular B.*

(ii) Moreover, the bijection $\pi(B,BG) \to St(B,G)$ is established by the operation $f \mapsto f^!p$ of inducing, i.e. every principal G-bundle over a cell space B is equivalent to the bundle induced from the universal one by a map $f : B \to BG$, and two induced bundles are equivalent if and only if the inducing maps are homotopic.

(iii) A principal G-bundle is universal if and only if its total space is weakly contractible (i.e. WHE to a point).

(iv) The classifying space *BG is unique up to weak homotopy equivalence.*

The last statement is obvious, because for two classifying spaces BG and $\widetilde{BG}$, the bijection between $\pi(B,BG)$ and $\pi(B,\widetilde{BG})$ for every CW-complex B is obtained by the identification of each with $St(B,G)$. This identification is natural because $\varphi : B' \to B$ defines $\varphi^! : St(B,G) \to St(B',G)$ such that $\varphi^!(f^!p) = (\varphi^* f)^!p$ for all $f : B \to BG$.

D. Milnor's construction. It supplies a principal G-bundle whose total space is weakly contractible. Take EG to be the infinite join $G * G * G * \cdots$ equipped with the simultaneous action of G via right translations, and set $BG := EG/G$.

More explicitly, EG can be considered as a subset in the product of countably many copies of the cone CG. A point in the product is a sequence $(t_1,g_1),\dots,(t_i,g_i),\dots$ of pairs, where $t_i \in [0,1]$ is non-zero only for finitely many i, and $g_i \in G$, though g_i is relevant only when $t_k \neq 0$. The space EG is the subset in this product given by the equation $\sum t_i = 1$. On this set, the group G acts freely by $(\dots,(t_i,g_i),\dots) \mapsto (\dots,(t_i,g_i g),\dots)$, and $BG = EG/G$.

Note that open G-invariant subsets $E_kG \subset EG$ defined by $t_k \neq 0$ form an open cover of EG, and each projection $E_kG \to B_kG := E_kG/G$ comes with a section defined by $g_k = e$. Thus, B_kG form an open cover of BG equipped with local trivializations. This makes $EG \to BG$ a principal G-bundle. To show that EG is weakly contractible, we note that every spheroid in EG lands in a finite join G^{*n}, and is contractible in $G^{*(n+1)}$ which contains a cone $C(G^{*n})$.

Perhaps the only example of Milnor's construction which looks familiar rather than wild is the case of $G = \{\pm 1\} \cong \mathbb{Z}_2$. The cone CG can be identified with the interval $[-1,1]$, and EG with the ∞-dimensional sphere $\sum t_i^2 = 1$ in $[-1,1]^\infty$ (homeomorphic to the "∞-dimensional diamond" $\sum |t_i| = 1$). Thus, $B\mathbb{Z}_2 = \mathbb{R}P^\infty = S^\infty/(\pm 1)$.

E. Proof of the classification theorem. We begin with a lemma *à la Feldbau's*:

`Lemma.` *A principal G-bundle over a cube I^n is trivial.*

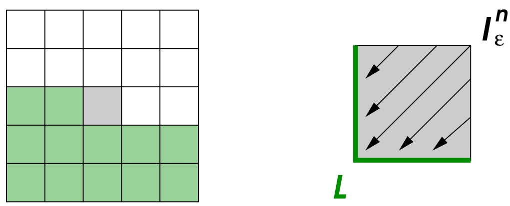

Figure 47: G-bundles over I^n are trivial

`Proof.` Partitioning I^n into sufficiently many sufficiently small cubes I^n_ε, we may assume that the bundle is trivial over each small cube. To trivialize the bundle over I^n we construct a section of it by inductively and systematically extending the previously constructed section over a part of I^n (shown green in Figure 47) to one extra cube at a time. At a step of induction, we have the section defined on a part L of the boundary ∂I^n_ε, and we want to extend it to the whole of I^n_ε. But the section of a trivial bundle is a function from the base to G. So, we can extend $L \to G$ to $I^n_\alpha \to G$ by composing the former with a retraction of I^n_ε to L.

`Theorem.` *Principal G-bundles over a cellular base, induced from the same bundle by homotopic maps, are equivalent.*

`Proof.` Let Z be a CW-complex, and (f_t) a homotopy between $f_0, f_1 : Z \to B$. Consider two bundles over $Z \times I$, both induced from the same principal G-bundle $p : E \to B$: one by (f_t), the other by $f_0 \times \mathrm{id}_I$. They are identified over $Z \times 0$, and we want to extend this identification (equivalence) to the whole of $Z \times I$. Restricted to $Z \times 1$, it will provide an equivalence between $f_0^! p$ and $f_1^! p$.

The method is cell induction. At a step of it, we have the bundles identified over $(Z \times 0) \cup (Z^{n-1} \times I)$, and we extend this identification to $(Z \times 0) \cup (Z^n \times I)$ one n-cell of Z at a time. Thus, over the "filled glass" $D^n \times I$, we have two bundles (induced by the characteristic map $D^n \times I \to Z \times I$ from those on $Z \times I$), which are both trivial by above Lemma, and which are already identified over the "empty glass" $(D^n \times 0) \cup (\partial D^n \times I)$.

Note that a G-equivariant equivalence of trivial bundles is given by a function $x \mapsto s(x)$ from the base to the group: $(x, g) \mapsto (x, s(x)g)$. Thus, we need to extend such a function from the "empty glass" to the "filled glass". This is done by Borsuk's retraction of the latter to the former. $\square$

Thus, for a cellular B, we have a map $\pi(B, BG) \to St(B, G)$ well-defined by inducing. The following proposition applied to CW-pairs $(Z, W) = (B, \emptyset)$ and $(Z, W) = (B \times I, B \times \partial I)$ proves that it is surjective and injective respectively.

`Proposition.` *Let $p : EG \to BG$ be a principal G-bundle such that $\pi_k(EG) = 0$ for all $k \geq 0$. Given a CW-pair (Z, W), a map $f : W \to BG$, and a principal G-bundle $q : E \to Z$, such that its restriction $q^{-1}(W) \to W$ to W coincides with $f^! p$, the bundle q can be induced from p by a map $F : Z \to BG$ such that $F|_W = f$:*

$$\begin{tikzcd}
E \arrow[d, "q"] \arrow[rr, bend left] & q^{-1}(W) \arrow[l, hook'] \arrow[r] \arrow[d] & EG \arrow[d, "p"] \\
Z \arrow[rr, bend right, "F"'] & W \arrow[l, hook'] \arrow[r, "f"] & BG
\end{tikzcd}$$

`Proof:` cell induction. By Lemma, the bundle, induced to D^n from q by the characteristic map of an n-cell, is trivial. So, we have a G-equivariant commutative diagram

$$\begin{tikzcd}
D^n \times G \arrow[d] & \partial D^n \times G \arrow[l, hook'] \arrow[r] \arrow[d] & EG \arrow[d, "p"] \\
D^n \arrow[urr, dashed] & \partial D^n \arrow[l, hook'] \arrow[r] \arrow[u, bend left, "e"] \arrow[ur] & BG
\end{tikzcd}$$

Note that inducing a trivial bundle amounts to a map from the base to EG (shown blue in the diagram and defined by the composition of the map between total spaces with the section $x \mapsto e \in G$ of the trivial bundle). Therefore our problem reduces to extending $\partial D^n \to EG$ to $D^n \to EG$, which is possible because $\pi_{n-1}(EG) = 0$. $\square$

Thus, we have proved statements (i) and (ii) of the classification theorem (based on Milnor's bundle $p : EG \to BG$), and also found that weak contractibility of the total space is sufficient for a principal G-bundle to be universal. To show that this condition is necessary, we take a cellular approximation $f : B \to BG$ of Milnor's classifying space, and examine the induced bundle $f^! p : E \to B$. The map between the bundles yields a morphism of exact homotopy sequences:

$$\begin{array}{ccccccccc}
\pi_{k-1}(G) & \leftarrow & \pi_k(B) & \leftarrow & \pi_k(E) & \leftarrow & \pi_k(G) & \leftarrow & \pi_{k+1}(B) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{k-1}(G) & \leftarrow & \pi_k(BG) & \leftarrow & \pi_k(EG) & \leftarrow & \pi_k(G) & \leftarrow & \pi_{k+1}(BG)
\end{array}$$

Since $B \to BG$ is a WHE, the four black vertical arrows are isomorphisms, and the 5-lemma implies that $\pi_k(E) = \pi_k(EG) = 0$. Consequently, $f^! p : E \to B$ is universal.

Consider now any universal principal G-bundle $\widetilde{p} : \widetilde{EG} \to \widetilde{BG}$. Since B is cellular, $f^! p$ can be induced from $\widetilde{p}$ by a map $\widetilde{f} : B \to \widetilde{BG}$. So, we have a commutative ladder as above with $\widetilde{EG}$ and $\widetilde{BG}$ instead of Milnor's EG and BG. Yet, since both bundles $\widetilde{p}$ and $\widetilde{f}^! \widetilde{p}$ are universal, the inducing map $\widetilde{f}$ is a WHE (as we have observed in our argument establishing part (iv) of the classification theorem). Therefore $\pi_k(\widetilde{EG}) = \pi_k(E) = 0$.

This completes the proof of the classification theorem.

`Corollary.` *Let $H \subset G$ be a subgroup such that the inclusion is a WHE. Then for any cell space B, the map $St(B, H) \to St(B, G)$ defined by the expansion of the structure group is bijective.*

`Proof.` Expand the structure group of a universal H-bundle $EH \to BH$ from H to G by interpreting H-valued clutching functions as G-valued. They define a principal G-bundle $E \to BH$ and a fiberwise inclusion:

$$\begin{tikzcd}
EH \arrow[rr, hook] \arrow[dr, "H"'] & & E \arrow[dl, "G"] \\
& BH &
\end{tikzcd}$$

In fact $E \to BH$ is the same as the associated H-bundle with the fiber G defined by left translations of H on G. The maps between the fibers coincide with the embedding $H \subset G$ up to left and/or right translations by H, and so they are weak homotopy equivalences. From the morphism of exact homotopy sequences of these fibrations we conclude (invoking the 5-lemma as above) that the embedding $EH \subset E$ is a WHE. Consequently, E is weakly contractible, $E \to BH$ is a universal G-bundle, and $St(B, G) = \pi(B, BH) = St(B, BH)$ for any cell space B.

EXERCISES

111. For a smooth free action of a compact Lie group G on a smooth manifold M, prove that M/G is a smooth manifold, and the canonical projection $M \to M/G$ is a smooth locally trivial bundle.

112. Given a principal G-bundle, consider the associated bundle with the fiber G defined by the adjoint action of G on itself. Show that the fibers of this associated bundle carry a group structure isomorphic to that of G, and respectively all sections of the associated bundle form a group with respect to pointwise multiplication. (It is called the *gauge group* of the given principal G-bundle.)

113. Show that the bundle $\pi : T \to B$ with the fiber G, associated with a principal G-bundle $p : E \to B$ and the action of G on itself by left translations, is canonically identified with original bundle p. Yet, can you explain where the structure of π as a principal bundle (i.e. the action of G on T) comes from?

114. Let $p : E \to B$ be a principal G-bundle. Show that the induced bundle $p^! p$ is trivial. More generally, let $H \subset G$ be a subgroup, $q : E \to E/H$ the principal H-bundle, and $\pi : E/H \to B$ the bundle with the fiber G/H such that $\pi \circ q = p$. Identify π with the bundle associated with p, $\pi^! \pi$ with the bundle associated with q, and show that the $\pi!\pi$ has a tautological section.

115. Let $\{\varphi_{\alpha\beta}\}$ and $\{\widetilde{\varphi}_{\alpha\beta}\}$ be two collections of clutching functions for the same open cover $B = \bigcup_\alpha U_\alpha$, defining two principal G-bundles. Show that the equivalence of these bundles is established by a collection of local re-trivializations (i.e. effectively by functions $h_\alpha : U_\alpha \to G$) such that $\widetilde{\varphi}_{\alpha\beta}(x) = \hbar_\beta(x)\varphi_{\alpha\beta}(x)h_\alpha^{-1}(x)$ for $x \in U_\alpha \cap U_\beta$.

116. Compute clutching functions in Milnor's construction.

117. Prove that $\pi_k(BG) = \pi_{k-1}(G)$ for all $k \geq 1$.

118. Show that for $n > 0$ and a path-connected G, $\pi_n(BG) = St(S^n, G) = \pi_{n-1}(G)$, and that the latter identification can be described as gluing a principal G-bundle over S^n from trivial bundles over two hemispheres by a single clutching function $S^{n-1} \to G$ on the equator $S^{n-1} \subset S^n$.

119. Is Lemma *à la Feldbau's* more general than Feldbau's lemma or a special case of it (for principal rather than all locally trivial bundles)?

120. For a closed Lie subgroup H in a Lie group G, show that the inclusion $H \hookrightarrow G$ is a WHE when G/H is contractible.

121. Prove that a continuous group homomorphism $\rho : G \to G'$ induces a map $BG \to BG'$ (at least when both classifying spaces are Milnor's or when BG is cellular) which is a WHE provided that ρ is.

Lecture 15. Classifying spaces

The criterion of weak contractibility of the total space for a principal G-bundle to be universal often allows one to replace Milnor's monstrous classifying space with a far handier model of BG.

A. Discrete groups. When G is discrete, the classifying space $BG = K(G,1)$, and the bundle $EG \to BG$ is a universal covering of $K(G,1)$. For the cellular model of $K(G,1)$, the universal covering space is also cellular and is therefore contractible.

B. Classical groups. Stiefel manifolds $V(\infty, n)$ are (weakly) contractible. Indeed, the fibration $V(\infty, n) \to S^\infty$ assigning to an n-frame its 1st vector has $V(\infty, n-1)$ as a fiber. Since S^∞ is contractible, the EHS of the fibration implies that for all $k \geq 0$

$$\pi_k(V(\infty, n)) = \pi_k(V(\infty, n-1)) = \cdots = \pi_k(V(\infty, 1)) = 0,$$

because $V(\infty, 1) = S^\infty$. The same argument works for $\mathbb{C}V(\infty, n)$ and $\mathbb{H}V(\infty, n)$. Therefore Stiefel fibrations

$$V(\infty, n) \overset{O_n}{\to} G(\infty, n), \quad V(\infty, n) \overset{SO_n}{\to} G_+(\infty, n)$$
$$\mathbb{C}V(\infty, n) \overset{U_n}{\to} \mathbb{C}G(\infty, n), \quad \mathbb{H}V(\infty, n) \overset{Sp_n}{\to} \mathbb{H}G(\infty, n)$$

are universal principal G-bundles for $G = O_n, SO_n, U_n$, and Sp_n.

In the case $O_1 \cong \mathbb{Z}_2$ this is Milnor's model $S^\infty \to \mathbb{R}P^\infty$. For U_1 and Sp_1, these are the Hopf bundle $S^\infty \to \mathbb{C}P^\infty$ and its quaternionic version $S^\infty \to \mathbb{H}P^\infty$.

Note that $\mathbb{C}P^\infty$ is fibered over $\mathbb{H}P^\infty$ with the fiber $Sp_1/U_1 = \mathbb{C}P^1$. This illustrates the general rule: $EG \to EG/H$, where H is a subgroup of G, is a universal principal H-bundle (because EG is weakly contractible), and $BH = EG/H$ is fibered over BG with the fiber G/H.

As an important example, consider the maximal torus $T^n \subset U_n$ (it consists of diagonal unitary matrices). We obtain a map $BT^n = (\mathbb{C}P^\infty)^n \to \mathbb{C}G(\infty, n) = BU_n$ with the fiber $U_n/T_n = F_n(\mathbb{C})$. This should not be understood too literally: The universal T^n-bundle over $(\mathbb{C}P^\infty)^n$ is induced from the universal T^n-bundle over $\mathbb{C}V(\infty, n)/T^n$ by a map which is a WHE, and it is the latter space which is fibered over $\mathbb{C}G(\infty, n)$ with the fiber $F_n(\mathbb{C})$.

Similarly, there are homotopy unique maps $(\mathbb{R}P^\infty)^n \to G(\infty, n)$ and $(\mathbb{H}P^\infty)^n \to \mathbb{H}G(\infty, n)$ with the homotopy fibers respectively $O_n/(\mathbb{Z}_2)^n = F_n(\mathbb{R})$ and $Sp_n/(Sp_1)^n = F_n(\mathbb{H})$.

When the subgroup $H \subset G$ is normal, the situation is more interesting. First, there is a map $\pi : BG \to B(G/H)$ defined by composing G-valued clutching functions with the quotient homomorphism $G \to G/H$. The map induces from the universal G/H-bundle over $B(G/H)$ the associated G/H-bundle over BG with the structure group G. Then the following homotopy commutative diagram (where, in the spirit of Remark at the end of Lecture 13, pt stands for $E(G/H)$) shows that the homotopy fiber of π is BH:

$$\begin{tikzcd} BH \arrow[r] \arrow[d, "G/H"'] & pt \arrow[d, "G/H"] \\ BG \arrow[r, "\pi"] & B(G/H) \end{tikzcd}$$

Here is an example corresponding to the determinant homomorphism $\det : U_n \to U_1$:

$$\begin{tikzcd} BSU_n \arrow[r] \arrow[d, "U_1"'] & pt \arrow[d, "U_1"] \\ \mathbb{C}G(\infty, n) \arrow[r] & \mathbb{C}P^\infty \end{tikzcd}$$

C. Classification of vector bundles. According to our general theory, classification of vector bundles over a given cellular base is equivalent to the classification of principal G-bundles (with the appropriate group G) over the same base. For n-dimensional real, complex, or quaternionic vector bundles, the structure groups are $GL_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ respectively. The grassmannians $G(\infty, n), \mathbb{C}G(\infty, n)$ and $\mathbb{H}G(\infty, n)$ serve as the respective classifying spaces because, as we noted in Lecture 14, the groups $GL_n(\mathbb{K})$ are homotopy equivalent to their respective maximal compact subgroup O_n, U_n, Sp_n. The grassmannians $G_+(\infty, n)$ serve as classifying spaces for *oriented* n-dimensional vector bundles, because the subgroup $GL_n^+(\mathbb{R})$ of matrices with $\det > 0$ Gram-Schmidt retracts to SO_n (or because SO_n is the kernel of $\det : O_n \to O_1$).

In each of the 4 cases (real, complex, quaternionic, real-oriented), the vector bundle over the classifying space associated with the standard vector representation of $GL_n(\mathbb{K})$ (or $GL_n^+(\mathbb{R})$) on $\mathbb{K}^n$ is the *tautological vector bundle* of the grassmannian: The fiber of it over a point in the grassmannian represented by an n-dimensional $\mathbb{K}$-subspace $V^n \subset \mathbb{K}^\infty$ is the space V^n itself. Thus, these tautological bundles play the role of universal vector bundles over the classifying spaces:

`Theorem.` *The set of equivalence classes of real, real oriented, complex or quaternionic vector bundles of dimension n over a given CW-complex B coincides respectively with*

$$\pi(B, G(\infty, n)),\ \pi(B, G_+(\infty, n)),\ \pi(B, \mathbb{C}G(\infty, n)),\ \pi(B, \mathbb{H}G(\infty, n)).$$

Namely, every n-dimensional vector bundle over B is equivalent to a vector bundle induced from the tautological one by a map to the classifying space, and the vector bundles induced by homotopic maps are equivalent (and vice versa*).*

`Remark.` When B is a CW-complex of finite dimension $< N$, the set $\pi(B, G(\infty, n))$ depends only on the N-dimensional skeleton of the classifying space and hence coincides with $\pi(B, G(N+n, n)$. Similar estimates in the complex and quaternionic cases are even more encouraging. Ultimately our classification theory makes a lot of sense: The most topologically complicated (finite-parametric) families of n-dimensional vector spaces are the families of *all* n-dimensional subspaces in coordinate spaces of sufficiently high dimensions.

`Examples.` (a) Contractible maps to the classifying space induce trivial bundles, and *vice versa*. Consequently, complex vector bundles over S^1, as well as quaternionic vector bundles over any CW-complex of dim ≤ 3, are trivial: The classifying grassmannians don't have cells in dimensions 1 and $1, 2, 3$ respectively.

(b) For all $n \geq 1$, $\pi_1(G(\infty, n)) = \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$. Consequently, up to equivalence, there is only one non-trivial real line bundle over S^1: the tautological (*Möbius*) line bundle over $\mathbb{R}P^1$, i.e. the family of all 1-dimensional subspaces in $\mathbb{R}^2$. (Why is it called "Möbius"?) Moreover, every real vector bundle over S^1 is either trivial, or the direct sum of a trivial bundle with the Möbius line bundle.

(c) $\pi_2(\mathbb{C}G(\infty, n)) = \pi_2(sk_3(\mathbb{C}G(\infty, n)) = \pi_2(\mathbb{C}P^1) = \mathbb{Z}$. Consequently, every complex vector bundle over $\mathbb{C}P^1$ is equivalent to the direct sum of a trivial bundle with a complex line bundle induced from the tautological (Hopf) line bundle L over $\mathbb{C}P^1$ by a degree-d map $\mathbb{C}P^1 \to \mathbb{C}P^1$. We leave it as an exercise for the reader to check that the induced line bundle is equivalent to $L^{\otimes d}$.

D. K-functor. Here we will consider complex (for the sake of definiteness) vector bundles over a base X, with the dimensions of the fibers over different path-connected components of the base allowed to be different. As before, two such vector bundles are equivalent if there is a fiberwise linear homeomorphism between their total spaces which induces the identity map of the base. The set of equivalence classes forms an abelian semigroup with respect to the operation of the direct sum of vector bundles.

To every nonempty abelian semigroup, Grothendieck's *K-functor* associates a homomorphism $S \to K(S)$ from that semigroup to a certain abelian group with the property that every homomorphism $S \to A$ to an abelian group factors canonically through $K(S)$:

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & K(S) \arrow[dl, dashed] \\ & A & \end{tikzcd}$$

The construction of $K(S)$ essentially mimics the way one constructs integers from natural numbers. Consider ordered pairs $a \ominus b$ of elements of S (where the operation is denoted by $\oplus$), and introduce an equivalence relation: $a \ominus b \sim c \ominus d$ whenever $a \oplus d \oplus e = b \oplus c \oplus e$ for some $e \in S$. (In a semigroup like $\mathbb{N}$: with the cancellation law, adding e to both sides would be redundant.) Then $K(S)$ is defined as the set of equivalence classes equipped with the operation $(a \ominus b) \oplus (c \ominus d) := (a \oplus c) \ominus (b \oplus d)$. The map $S \to K(S)$ is well-defined by $s \mapsto (s \oplus e) \ominus e$ for any $e \in S$. Checking the correctness of these definitions is straightforward (and the universality property is obvious).

Applying the K-functor to the above semigroup, we obtain the Grothendieck group of complex vector bundles over X, which is denoted $K(X)$. Its elements are represented by *virtual* vector bundles $\xi \ominus \eta$. In fact $K(X)$ is a commutative ring with respect to the operation of tensor product of (virtual) vector bundles. This is easy to verify starting with $(\xi \ominus \eta) \otimes \zeta \sim (\xi \otimes \zeta) \ominus (\eta \otimes \zeta)$. The trivial one-dimensional vector bundle plays the role of the unit element.

`Example.` $K(pt) = \mathbb{Z}$ since a vector bundle over a point is just a vector space, its dimension is the only invariant, and it is additive and multiplicative relative to the operations of direct sum and tensor product respectively.

E. Stable equivalence. We will see here that the ring $K(X)$ of a *finite* CW-complex X is a homotopy invariant.

Two vector bundles are called *stably equivalent* if they become equivalent after adding to them trivial bundles of suitable (possibly different) dimensions. For instance, the normal bundle to the standard sphere $S^2 \subset \mathbb{R}^3$ is trivial, and its sum with the tangent bundle to S^2 is $\mathbb{R}^3$, i.e. also trivial. Thus, the tangent bundle to S^2 is *stably trivial* (yet non-trivial — prove it though). This applies to the complexifications of these bundles as well.

Let X be a path-connected cell space (so that complex vector bundles over X have definite dimensions). A bundle ξ is induced from the tautological one by a map $f : X \to \mathbb{C}G(\infty, n)$ with $n = \dim \xi$. Stabilizations $\xi \oplus \mathbb{C}^k$ are induced from the tautological bundle over $\mathbb{C}G(\infty, n+k)$ by the composition of f with the embeddings $\mathbb{C}G(\infty, n) \hookrightarrow \mathbb{C}G(\infty, n+k)$ defined by adding the same $\mathbb{C}^k$ to both: a subspace $V^n \subset \mathbb{C}^\infty$ representing a point in the grassmannian and the ambient space $\mathbb{C}^\infty$. Two bundles ξ and η of dimensions n and m are stably equivalent if and only if such composite inducing maps to $\mathbb{C}G(\infty, N)$ for sufficiently large N are homotopic. Consequently, the set of classes of stable equivalence of complex vector bundles over X is the direct limit of $\pi(X, \mathbb{C}G(\infty, n))$ as $n \to \infty$.

The direct limit of the inclusion sequence

$$\mathbb{C}G(\infty, N) \subset \mathbb{C}G(\infty, N+1) \subset \mathbb{C}G(\infty, N+2) \subset \cdots$$

is denoted BU. The (weak) homotopy type of BU can be described abstractly as the direct limit of BU_N under the sequence of maps $BU_N \to BU_{N+1}$ defined by the standard inclusions of U_N into U_{N+1}, or as the classifying space for principal bundles with the structure group $U := \varinjlim U_N$. While $\varinjlim \pi(X, BU_N)$ is a subset in $\pi(X, BU)$, the equality is not guaranteed unless X is finite dimensional.

`Proposition.` *The set of classes of stable equivalence of complex vector bundles over a path-connected finite dimensional CW-complex X coincides with $\pi(X, BU)$.*

Indeed, Schubert cell partitions of the grassmannians $\mathbb{C}G(\infty, N)$ are compatible with the above inclusions and define a CW-structure on BU. By the cellular approximation theorem, if $\dim X < n$, a map $X \to BU$ is homotopic to some map $X \to \mathbb{C}G(n, [n/2]) \subset BU$.

`Lemma.` *Let X be a finite CW-complex (not necessarily path-connected). For every complex vector bundle ξ over X (possibly of different dimensions over different path-connected components of X) there exists a complex vector bundle $\xi^\perp$ such that $\xi \oplus \xi^\perp$ is trivial.*

`Proof.` The restrictions $\xi|_{X_i}$ (of dimensions n_i) to the (finitely many) path-connected components X_i of X are induced from the tautological bundles by some maps $f_i : X_i \to \mathbb{C}G(N, n_i)$ provided that N is large enough. The fibers $V^{n_i} \subset \mathbb{C}^N$ of the tautological bundle have Hermitian orthogonal complements which are the fibers of the other tautological bundle over $\mathbb{C}G(N, n_i) = \mathbb{C}G(N, N - n_i)$. Let $\xi^\perp|_{X_i}$ be induced from it by the same f_i. Then $\xi \oplus \xi^\perp = \mathbb{C}^N$.

`Theorem.` *For a finite CW-complex X, $K(X) = \pi(X, \mathbb{Z} \times BU)$.*

`Proof.` It suffices to consider the case of a path-connected X. Then $\dim(\xi \ominus \eta) := \dim \xi - \dim \eta$ is well-defined as a (ring) homomorphism $K(X) \to \mathbb{Z}$ (if you wish, induced by the restriction of vector bundles to a point in X). On the other hand, $\xi \ominus \eta \sim (\xi \oplus \eta^\perp) \ominus N$, where N is the favorite notation of topologists for the trivial N-dimensional vector bundle $\mathbb{C}^N$, $\eta \oplus \eta^\perp$ in this case. We claim that the class of $\xi \ominus \eta$ in $K(X)$ is uniquely determined by its dimension together with the class of stable equivalence of the bundle $\xi \oplus \eta^\perp$, and *vice versa*. Indeed, $\xi \ominus \eta \sim \widetilde{\xi} \ominus \widetilde{\eta}$ means that for some ζ

$$\xi \oplus \widetilde{\eta} \oplus \zeta \oplus \eta^\perp \oplus \widetilde{\eta}^\perp \oplus \zeta^\perp \sim \widetilde{\xi} \oplus \eta \oplus \zeta \oplus \eta^\perp \oplus \widetilde{\eta}^\perp \oplus \zeta^\perp$$

i.e. that $\xi \oplus \eta^\perp$ is stably equivalent to $\widetilde{\xi} \oplus \widetilde{\eta}^\perp$. Conversely, from

$$\xi \oplus \eta^\perp \oplus M \oplus \eta \oplus \widetilde{\eta} \sim \widetilde{\xi} \oplus \widetilde{\eta}^\perp \oplus \widetilde{M} \oplus \eta \oplus \widetilde{\eta}$$

it follows that $\xi \oplus \widetilde{\eta} \oplus (M + N) \sim \widetilde{\xi} \oplus \eta \oplus (\widetilde{M} + \widetilde{N})$, where $M + N = \widetilde{M} + \widetilde{N}$ provided that $\dim \xi - \dim \eta = \dim \widetilde{\xi} - \dim \widetilde{\eta}$.

Thus, for a path-connected X, a class of $K(X)$ corresponds to a map of X to $\mathbb{Z}$ (the dimension of $\xi \ominus \eta$) and a homotopy class of maps $X \to BU$ (by Proposition).

`Remarks.` (1) The ring structure of $\pi(X, \mathbb{Z} \times BU) = \pi(X, \mathbb{Z}) \times \pi(X, BU)$ is induced by the usual operations in $\mathbb{Z}$, and by the maps $BU_N \times BU_M \to BU_{M+N}$ and $BU_N \times BU_M \to BU_{MN}$ (corresponding to the direct sum and tensor product of tautological vector bundles, or, equivalently, to the respective embeddings of $U_N \times U_M \subset U_{M+N}$ and $U_N \times U_M \subset U_{NM}$) in the limit $N, M \to \infty$.

(2) This result is the starting point of *complex K-theory*. We are not going to study it in this book, but eventually we will be able to identify the place of K-theory among generalized cohomology theories.

EXERCISES

122. For any path-connected CW-complex X, construct a map from X to (say, cellular) $K(\pi_1(X), 1)$ which induced an isomorphism of fundamental groups, and derive from this that X has a universal covering.

`Remark.` In fact (though we didn't prove this) CW-complexes are locally contractible (see e.g. [8]), and so they have universal coverings by our criterion from Lecture 7.

123. Construct a map $G(\infty, n) \to \mathbb{R}P^\infty$ with the homotopy fiber $G_+(\infty, n)$.

124. Classify principal $SL_2(\mathbb{C})$-bundles over $\mathbb{C}P^2$.

125. Prove that the canonical embedding $\mathbb{C}P^\infty \subset G_+(\infty, 2)$ (defined by considering a complex line in $\mathbb{C}^\infty$ as a real plane equipped with the complex orientation) is a homotopy equivalence.

126. Show that real line bundles over a path-connected cellular base B (and in fact any base which has a universal covering) are classified by elements of $Hom(\pi_1(B), \mathbb{Z}_2)$.

127. Show that the complex line bundle induced from the Hopf line bundle L over $\mathbb{C}P^1$ by a degree-d map $\mathbb{C}P^1 \to \mathbb{C}P^1$ is equivalent to $L^{\otimes d}$.
`Hint:` Use clutching functions.

128. Classify complex line bundles over $\mathbb{R}P^2$. (`Hint:` the same.)

129. Show that in the semigroups of the equivalence classes of (real or complex) vector bundles, the cancellation property can fail.

130. The semigroup of "material points" in a vector space V consists of pairs $[m, v]$ where $m > 0$ is the mass of point $v \in V$, while the operation $[m, v] + [m', v']$ yields the center of mass: $[m + m', (mv + m'v')/(m + m')]$. Compute Grothendieck's K-functor of this semigroup.

131. Show that inducing complex vector bundles by a map $f : X \to Y$ defines a ring homomorphism $K(Y) \to K(X)$.

132. Prove that for every group G, the loop space ΩBG is weakly homotopy equivalent to G.

`Remark.` Thus, from the homotopy theoretic point of view, ΩBG is indistinguishable from G and should be considered as a "group-like" object in the homotopy category. The official name for such a group-like object is *H-space*. By definition it is a space X equipped with *multiplication* map $\mu : X \times X \to X$, *inversion* map $\nu : X \to X$, and the *unit* $x^0 \in X$ such that: (i) $x \mapsto \mu(x, x^0)$ and $x \mapsto \mu(x^0, x)$ are homotopic to id_X, (ii) $x \mapsto \mu(x, \nu(x))$ and $x \mapsto \mu(\nu(x), x)$ are homotopic to the constant map $x \mapsto x^0$, and (iii) $\mu \times \mathrm{id}_X$ and $\mathrm{id}_X \times \mu$ are homotopic maps $X \times X \times X \to X$ (homotopy associativity of μ). Therefore, for every path-connected B, the loop space ΩB is an H-space (with μ defined by the composition of loops and ν by their reverse parameterization), while the Hurewicz fibrations $pt \overset{\Omega B}{\to} B$ generalize universal principal G-bundles.

Lecture 21. Thom's transversality theorem

We review here some analysis on manifolds necessary for forthcoming applications (and refer to [6] for further details and developments).

A. Jet spaces. Let $\pi : E \to B$ be a smooth bundle, by which we mean an infinitely differentiable submersion of C^∞-manifolds which possesses local C^∞-trivializations. Two sections f and g are said to have the same *k-jet* at $x \in B$ (notation: $j_x^k f = j_x^k g$) if they have the same value y at x and the same Taylor coefficients of orders $\leq k$ in some (and hence any) local chart and trivialization near (x, y).

The space of k-jets of sections at all $x \in B$ is denoted $J^k(\pi)$ (or $J^k(B, F)$ when the bundle is trivial so that the sections are just smooth maps $B \to F$, and simply $J^k(B)$ in the case of smooth functions $B \to \mathbb{R}$). It is a manifold (with transition maps between charts determined by the behavior of Taylor coefficients under changes of coordinates), which fits in the infinite tower of affine fibrations

$$\cdots \to J^k(\pi) \to J^{k-1}(\pi) \to \cdots \to J^1(\pi) \to J^0(\pi) = E \xrightarrow{\pi} B$$

with the fiber over $j_x^{k-1} f$ isomorphic to $T_{f(x)}F \otimes S^k T_x^* B$.

A section $f \in C^\infty(\pi)$ comes with its *k-jet extension* $x \mapsto j_x^k f$: a section $j^k f : B \to J^k(\pi)$ of the k-jet fibration $J^k(\pi) \to B$. It is integral (in the sense of Frobenius' integrability theorem) to the so-called *Cartan distribution*. Namely, in local components $f = (f_1, \dots, f_n)$ and local coordinates $x = (x_1, \dots, x_m)$ we have:

$$d\left(\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}\right) = \sum_{j=1}^{m} \frac{\partial^{|\alpha|+1} f_i}{\partial x^{\alpha+1_j}} dx_j, \ \ (\alpha + 1_j := (\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_m)).$$

The Cartan distribution on $J^k(\pi)$ is given by the following system of linear equations on tangent spaces to $J^k(\pi)$:

$$dp_i^\alpha = \sum_j p_i^{\alpha+1_j} dx_j, \quad 0 \leq |\alpha| := \sum_j \alpha_j < k, \quad i = 1, \dots, n,$$

where p_i^α are the namesakes of partial derivatives of f_i considered as independent local coordinates on the fiber of the jet fibration.

`Example.` On the space $J^1(B) = \mathbb{R} \times T^* B$ of 1-jets of functions, the Cartan distribution is the famous *contact structure* — the "maximally non-integrable" field of tangent hyperplanes $du = \sum p^j dx_j$.

B. C^∞-topology. The space $C^\infty(\pi)$ of smooth sections of a smooth bundle $\pi : E \to B$ is embedded into the space of continuous sections $B \to J^k(\pi)$ of the k-jet fibration, and inherits from it the compact-open topology (called C^k-topology). The C^∞-topology on $C^\infty(\pi)$ is defined as the union of C^k-topologies over all $k \geq 0$. In fact this is the topology of uniform convergence of sections and all their derivatives on compact subsets of B, and our current goal is to turn $C^\infty(\pi)$ into a complete metric space.

To this end, note that every manifold M can be equipped with a Riemannian metric, making it a metric space (with the distance between two points in the same connected component defined as the infimum of the lengths of all smooth curves connecting them). The latter can be turned into a complete metric space by embedding M into the cylinder $M \times \mathbb{R}$ as the graph of a continuous function $M \to \mathbb{R}$ which tends to $+\infty$ "at infinity" of the one-point compactification of M, and by inducing the product metric from the cylinder.

Given a complete metric d_k on $J^k(\pi)$, and a compact $K \subset B$, put

$$D_{k,K}(f,g) := \max_{x\in K} d_k(j^k_x f, j^k_x g).$$

This is a semi-distance on $C^\infty(\pi)$ in the sense that it is non-negative, satisfies the triangle inequality, but can vanish even if $f \neq g$. Doing this for every k and for a sequence of compact subsets covering B, we obtain a countable sequence D_i of such semi-distances on $C^\infty(\pi)$. One turns it into the *Fréchet metric*

$$D(f,g) := \sum_i \frac{1}{2^i} \frac{D_i(f,g)}{1 + D_i(f,g)}.$$

A sequence of sections $f_n \in C^\infty(\pi)$ is Cauchy relative to D if and only if it is Cauchy with respect to each D_i i.e. effectively if for each $k \geq 0$ and each compact $K \subset B$, the sequence $j^k_x f_n \in J^k(\pi)$ is Cauchy with respect to d_k uniformly over $x \in K$.

From the completeness of spaces of continuous functions on compact sets under uniform convergence, we conclude that $j^k f_n$ converge uniformly on compact subsets to some continuous sections $g^{(k)} : B \to J^k(\pi)$. To see that $g^{(k)}$ are the k-jet extensions of $g^{(0)}$, recall that $j^{(k-1)} f_n$ can be recovered from $j^k f_n$ by integration along curves in B, and that such integration commutes with uniform limits. Therefore, each $g^{(k-1)}$ is obtained from $g^{(k)}$ this way, making it C^1. By induction on k, $g^{(0)} \in C^\infty(\pi)$ with $j^k g^{(0)} = g^{(k)}$ for all k. Thus, $C^\infty(\pi)$ equipped with the Fréchet metric D is complete.

`Remark.` An order k *partial differential relation* (e.g. a PDE system) on sections of a bundle π is an "algebraic" relation between partial derivatives, i.e. it is defined by a subset $R \subset J^k(\pi)$. The space of solutions consists of sections whose k-jet extensions land in R. It is included into the space of *formal* solutions: sections $B \to J^k(\pi)$ landing in R but not necessarily integral to the Cartan distribution. It turns out that under some condition on R, the inclusion is a weak homotopy equivalence (a property named *h-principle* — for "homotopy", but probably also after M. Hirsch who established it for immersions). Powerful sufficient conditions for this property (e.g. it turns out to hold for any partial differential relation invariant under diffeomorphisms of B whenever all components of B are non-compact) are proved in the book [7] by M. Gromov, which provides a broad range of applications of the h-principle. Note that the coincidence of π_0 alone reduces the existence problem for solutions of differential relations to a purely topological question.

C. Massive subsets. The complement to a (say, rational) point in $\mathbb{R}$ is open and dense. The complement to all rational points is not open but still dense. This illustrates

`Baire's theorem`. *In a complete metric space, intersections of countably many open dense sets are dense.*

`Proof.` Inside an open ball B centered at a, pick a_1 from the 1st open dense set U_1 lying in $B \cap U_1$ together with a smaller closed ball $\overline{B}_1$ at most half the radius of B; inside B_1 pick a_2 from the 2nd open dense set U_2 lying in $B_1 \cap U_2$ together with a closed ball $\overline{B}_2$ at most half the radius of B_1; and so on. The sequence $\{a_n\}$ is Cauchy and its limit lies in $B \bigcap_{i=1}^{\infty} U_i$. $\square$

Subsets of a complete metric space representable as intersections of countably many open dense sets are called *massive*. Clearly, intersections of countably many massive sets is massive, and by Baire's theorem dense — hence certainly non-empty. In contrast, taking from $\mathbb{Q} \subset \mathbb{R}$ one rational point after another, we can obtain a sequence of dense sets in $\mathbb{R}$ whose intersection is empty.

In applications to spaces of sections $C^\infty(\pi)$, whenever one says that a certain property of sections is "typical", or that a "generic" section has this property, one means that the sections possessing the property form a massive subset in $C^\infty(\pi)$. In particular, in any C^∞-neighborhood of a given section there are sections which have the property, or equivalently, that even if a given section doesn't have it, it can always be "perturbed" into one that does.

D. Transversality. A smooth map $f : M \to N$ is called *transverse* to a submanifold $Z \subset M$ (which is written as $f \pitchfork Z$) if at every $x \in M$ such that $f(x) \in Z$ the image of the tangent space to M at x under the differential of f at x together with the tangent space to Z at $f(x)$ span the tangent space to N at $f(x)$:

$$d_x f(T_x M) + T_{f(x)} Z = T_{f(x)} N.$$

The latter equality is impossible when $\dim M + \dim Z < \dim N$, in which case the transversality condition means that $f(M)$ and Z are disjoint. When they are not disjoint, $f \pitchfork Z$ implies that $f^{-1}(Z)$ is a submanifold in M of codimension equal to that of Z in N, and moreover *the normal bundle of $f^{-1}(Z)$ in M is induced by f from the normal bundle of Z in N:*

$$T_{f^{-1}(Z)} M \,/\, T(f^{-1}(Z)) = f^{!}\,(T_Z N \,/\, TZ).$$

Indeed, if f is given in local coordinates $(x_1, \dots, x_m)$ on M by $y_1 = f_1(x), \dots, y_n = f_n(x)$, and $y_1 = \dots = y_r = 0$ are local equations of Z in N, then $f^{-1}(Z)$ is locally given by the equations $f_1(x) = \dots = f_r(x) = 0$, and the transversality condition means that $dy_1 = df_1(x), \dots, dy_r = df_r(x)$ are pointwise linearly independent. By the Implicit Function Theorem, this guarantees that $f^{-1}(Z)$ is smooth and also shows that the linear coordinates $dy_1, \dots dy_r$ trivializing locally the normal bundle to Z in N also provide such a system of local linear coordinates on the normal bundle to $f^{-1}(Z)$ in M.

`Lemma.` *Suppose that the submanifold Z is a closed subset of N. Then transversality to Z at all points of a compact subset $K \subset M$ is an open condition in the C^1-topology of the space of smooth maps $f : M \to N$ (or sections of a bundle $N \to M$).*

`Proof.` It suffices to show that the set of 1-jets *non*-transverse to Z is closed in $J^1(M, N)$ (so that its complement is open). This set consists of triples (x, y, A) where $x \in M$, $y \in Z$, and $A \in Hom(T_x M, T_y N)$ is such that its composition with the projection $T_y N \to T_y N / T_y Z$ is not surjective. Let a sequence $(x^{(n)}, y^{(n)}, A^{(n)})$ of such triples converge to some $(x^{(0)}, y^{(0)}, A^{(0)})$. Then $y^{(0)}$ is still in Z since Z is assumed closed in N. In a local coordinate system near $(x^{(0)}, y^{(0)}) \in M \times N$ where Z is given by equations $y_1 = \dots = y_r = 0$, the non-transversality condition for an $n \times m$-matrix $A = (a_{ij})$ is that all its $r \times r$ minors in the top r rows vanish. This is a closed condition in the space of matrices, and if $A^{(n)}$ satisfy it for all n large enough, then so does their limit $A^{(0)}$. $\square$

`Thom's transversality theorem`. *Sections of a smooth bundle $\pi : E \to B$ whose k-jet extensions are transverse to a given submanifold $Z \subset J^k(\pi)$ form a massive subset in the C^∞-topology of $C^\infty(\pi)$.*

`Corollary` (elementary transversality theorem). *Smooth maps $f : M \to N$ transverse to a given submanifold $Z \subset N$ form a massive subset in the C^∞-topology of $C^\infty(M,N)$.*

`Remark.` One often considers (see [6]) the so-called *Whitney topology* (or "strong topology" which in our terminology is weaker than the C^∞-topology introduced above). Its base is formed by the sets of sections whose k-jet extensions land in a given open subset of $J^k(\pi)$ (everywhere over B, and not only over a given compact subset). The Whitney topology is metrizable only when B is compact (in which case it coincides with the C^∞-topology), but it still possesses the Baire property. It has the advantage that the above lemma remains true for $K = B$ even when B is non-compact, and consequently the massive set in Thom's transversality theorem is still open in the Whitney topology provided that Z is closed as a set in $J^k(\pi)$.

E. Reduction to Sard's lemma. To prove the theorem, we cover Z by countably many compact subsets Z_i each projecting to a compact subset $K_i \subset B$. By Baire's theorem and the above lemma, it suffices to show that every section f can be perturbed to make its k-jet extension transverse to Z_i (i.e. transverse to Z wherever $j^k_x f \in Z_i$). Moreover, we may assume that each Z_i is small enough so that its projection to $J^0(\pi) = E$ would fit in a product $\simeq \mathbb{R}^m \times \mathbb{R}^n$ of coordinate neighborhoods in a local trivialization of the bundle.

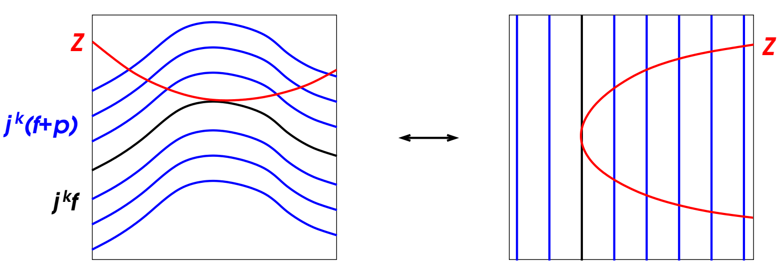

Figure 54: Fibration by graphs of $j^k(f+p)$

So, examine first our perturbation problem for $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ and $Z \subset J^k(\mathbb{R}^m, \mathbb{R}^n)$. Introduce the family $f(x) + p(x)$ of perturbations where the components $p_i(x) = \sum_{|\alpha| \le k|} p_i^\alpha x^\alpha$ of p are arbitrary polynomials in $x = (x_1, \dots, x_m)$ of degree $\le k$. The graphs of $j^k(f+p)$ are disjoint translates of the graph of $j^k f$, and fiber $J^k(\mathbb{R}^m, \mathbb{R}^n)$ over the space P of parameters p_i^α (Figure 54). The point now is that non-transversality of $j^k(f+p)$ to Z at x is equivalent to $j_x^k(f+p) \in Z$ being a critical point of the projection of $Z \to P$. By Sard's lemma (see [6] or [7] for a proof), the set of critical values (i.e. of those $p \in P$ for which $j^k(f+p)$ is non-transverse to Z) has zero measure in P, and so its complement is dense.

Our actual (global) problem differs from this model situation only because some values $f(x)$ for x within the local domain chart $\mathbb{R}^m$ might fall outside the codomain chart $\mathbb{R}^n$. To fix this, we pick two smooth compactly supported functions $\rho : \mathbb{R}^m \to \mathbb{R}$ and $\widetilde{\rho} : \mathbb{R}^n \to \mathbb{R}$ such that $\rho = 1$ on the projection of Z_i to $\mathbb{R}^m$, and $\widetilde{\rho} = 1$ on the projection of Z_i to $\mathbb{R}^n$. Then the perturbations $f(x) + \rho(x)\widetilde{\rho}(f(x))p(x)$ coincide with f outside the support of ρ and within this support wherever $f(x)$ falls outside the support of $\widetilde{\rho}$, and so they extend (by f) to global sections of $\pi : E \to B$. Yet, in a neighborhood in $\mathbb{R}^m \times \mathbb{R}^n$ containing the projection of Z_i, these perturbations coincide with $f+p$. In this neighborhood our previous arguments based on Sard's lemma apply, and provide global perturbations of f with k-jet extensions transverse to Z_i and as C^∞-close to f as desired. $\square$.

F. Applications. (1) A generic map $f : M \to M$ has only non-degenerate (and hence isolated) fixed points. To prove this, perturb the graph of f to make it transverse to the diagonal in $M \times M$.

(2) A generic *vector field* (i.e. a section of TM) as well as a generic differential 1-form (i.e. a section of T^*M) is transverse to the zero section, and so has only non-degenerate (and hence isolated) zeros.

(3) A generic smooth function $f : M \to \mathbb{R}$ is *Morse*, i.e. has only non-degenerate critical points: $\det(\partial^2 f/\partial x_i \partial x_j) \neq 0$ at every critical point. While (1) and (2) are applications of the elementary transversality theorem, here we use Thom's theorem applied to $Z := \mathbb{R} \times M \subset J^1(M) = \mathbb{R} \times T^*M$ (where $M \subset T^*M$ is the zero section).

EXERCISES

173. Explain why (3) does not follow from (2) even though the Morse condition is equivalent to *df* having only non-degenerate zeros.

174. Show that the assumption in Lemma that Z is closed is necessary.

175. Prove that immersions form a massive subset in $C^\infty(M^m, N^n)$ when $n \geq 2m$.

`Remark.` A "multi-jet" version of Thom's theorem (see [6]) implies that embeddings form a massive set in $C^\infty(M^m, N^n)$ when $n > 2m$.

Epilogue

Lecture 28. Spectra

Our achievements fit nicely into a broader picture of *extraordinary (co)homology theories*, which is outlined here with a view toward a next-level course in algebraic topology.

A. Eilenberg-Steenrod axioms. An abstract cohomology theory is a functor h from the category of *finite* CW-pairs to the category of $\mathbb{Z}$-graded abelian groups: To each finite CW-pair (X, A) it assigns a sequence $h^{\bullet}_{\bullet}(X, A)$ of abelian groups and to a continuous map $f : (X, A) \to (Y, B)$ a sequence of group homomorphisms $f_* : h_\bullet(X, A) \to h_\bullet(Y, B)$ (resp. $f^* : h^\bullet(Y, B) \to h^\bullet(X, A)$) compatible with compositions, which satisfy the *Eilenberg-Steenrod axioms*:

(1) Homotopy invariance: $f \sim g \;\Rightarrow\; f_* = g_*$ (resp. $f^* = g^*$).

(2) Factorization (excision/suspension): $h^{\bullet}_{\bullet}(X, A) = h^{\bullet}_{\bullet}(X/A, pt)$.

(3) Exactness: long exact sequences of CW-triples (X, A, B).
This includes existence of connecting homomorphisms $\partial_* : h_\bullet(X, A) \to h_{\bullet-1}(A, B)$ (resp. $\delta^* : h^\bullet(A, B) \to h^{\bullet+1}(X, A)$).

Reduced (co)homology is then defined by $\widetilde{h}^{\bullet}_{\bullet}(X) := h^{\bullet}_{\bullet}(X, x^0)$, and "non-reduced" by $h^{\bullet}_{\bullet}(X) := \widetilde{h}^{\bullet}_{\bullet}(X^+)$, where $X^+ := X \sqcup pt$.

Adding axiom

(4) Dimension: $h^{\bullet}_{\bullet}(pt) = 0$ for $\bullet \neq 0$,

one ends up with the ordinary cellular (co)homology theory with coefficients in $G = h_0(pt)$ (resp. $h^0(pt)$): Our argument in Section 17A of the coincidence between cellular and singular homology doesn't rely of anything but the axioms (1-4) and their corollaries, while in the case of infinite CW-complexes one also needs to require axiom

(5) Additivity: $h_\bullet(\bigsqcup_\alpha X_\alpha) = \bigoplus_\alpha h_\bullet(X_\alpha)$.

The point is, however, that by dropping axiom (4) one encounters many other, extraordinary homology and cohomology theories.

B. The Brown representability theorem says, loosely speaking, that all extraordinary cohomology theories come from spectra.

By definition, a *spectrum* $\mathcal{K}$ is a sequence of base point spaces K_n, $n = 0, 1, 2, \ldots$, and maps $f_n : \Sigma K_n \to K_{n+1}$, or equivalently $g_n : K_n \to \Omega K_{n+1}$. It is called an Ω-*spectrum* if g_n are homotopy equivalences (and a Σ-*spectrum* if f_n are, but as it will become clear shortly, this axiomatic definition is not particularly useful). In any case, all notions and notations in this theory are set in the homotopy category of base point spaces, and only homotopy types of K_n, f_n, and g_n are relevant.

Given a spectrum $\mathcal{K}$, one defines a cohomology theory "with coefficients in the spectrum" by

$$h^\bullet(X, A; \mathcal{K}) := \varinjlim_{N\to\infty} \pi(\Sigma^N(X/A), K_{\bullet+N}).$$

The direct limit is taken with respect to the compositions

$$\pi(\Sigma^N(X/A), K_{\bullet+N}) \xrightarrow{\Sigma} \pi(\Sigma^{N+1}(X/A), \Sigma K_{\bullet+N}) \xrightarrow{(f_{\bullet+N})_*} \pi(\Sigma^{N+1}(X/A), K_{\bullet+N+1})$$

or, equivalently,

$$\pi(\Sigma^N(X/A), K_{\bullet+N}) \xrightarrow{(g_{\bullet+N})_*} \pi(\Sigma^N(X/A), \Omega K_{\bullet+N+1}) = \pi(\Sigma^{N+1}(X/A), K_{\bullet+N+1}).$$

The structure of an abelian group is induced by the compositions of loops in ΩK_{n+1}. If $\mathcal{K}$ is an Ω-spectrum, the limit procedure is redundant since in this case $(g_n)_*$ are bijections. The corresponding homology theory "with coefficients in $\mathcal{K}$" is defined by direct limits

$$h_\bullet(X, A; \mathcal{K}) := \varinjlim_{N\to\infty} \pi(S^{\bullet+N}, K_N \# (X/A)) = \varinjlim_{N\to\infty} \pi_{\bullet+N}(K_N \# (X/A))$$

with respect to maps which should become obvious once we recall that $\Sigma K_N = S^1 \# K_N$ and that smash-product "#" is associative.

Finally, if $X = pt$, then $X^+ = S^0$, and so the *coefficient groups* of the theory $h_\bullet(pt) = h^{-\bullet}(pt) = \varinjlim \pi_{\bullet+N}(K_N)$ are the "stable homotopy groups" of the spectrum.

Note that the factorization axiom holds by the very construction, since only the quotient X/A of a pair (X, A) features in the definitions. The inherent properties (from Section 3B) of the bi-functor $\pi(-, -)$ guarantee the correct functoriality and homotopy invariance of our theories. So, the exactness axiom is the only one that needs explanations. For *co*homology theory, they come in the geometric form of the *Puppe sequence* (Figure 64):

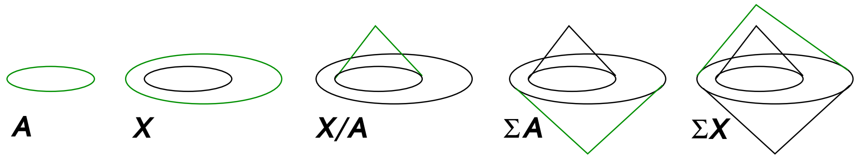

Figure 64: The Puppe sequence

This is a sequence of inclusions

$$A \subset X \subset X \cup CA \subset CX \cup CA \subset C(X \cup CA) \cup CX \subset \cdots$$

Here every 3 consecutive terms have the form $B \subset Y \subset Y \cup_B CB$, but when (X, A) is cellular, the sequence is homotopy equivalent to

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma X/\Sigma A \to \Sigma^2 A \to \dots$$

and so on.

For any K, consider the induced sequence

$$\pi(A, K) \leftarrow \pi(X, K) \leftarrow \pi(X/A, K) \leftarrow \pi(\Sigma A, K) \leftarrow \pi(\Sigma X, K) \leftarrow \dots$$

Recall that we work in the base point category where π means π_b, and each $\pi(Y, K)$ has a "zero" element represented by contractible maps. In this sense the sequence is exact: The kernel of each arrow coincides with the image of the previous one. Indeed, if the restriction $f|_B$ of a map $f : Y \to K$ is contractible, then f extends to a map $Y \cup_B CB \to K$, and *vice versa.*

For homology theory, there is a "dual Puppe sequence". One can start with turning an inclusion $B \subset Y$ into the homotopy equivalent Hurewicz fibration $E(Y, B) \to Y$. Its fiber F consists of paths in Y starting in B and terminating at $y^0 \in Y$. It is therefore fibered over B with the fiber ΩY. Iterating, we get the sequence of maps

$$Y \leftarrow B \leftarrow F \leftarrow \Omega Y \leftarrow \Omega B \leftarrow \Omega F \leftarrow \Omega^2 Y \leftarrow \cdots$$

where each space is homotopically the fiber of the Hurewicz fibration formed by the next two. Applying $\pi(Z, -)$ we get the sequence

$$\pi(Z, Y) \leftarrow \pi(Z, B) \leftarrow \pi(Z, F) \leftarrow \pi(\Sigma Z, Y) \leftarrow \pi(\Sigma Z, B) \leftarrow \cdots$$

which is exact due to the CHP of Hurewicz fibrations. With $Z = S^0$ we obtain the usual EHS of the pair (Y, B) since $\pi_n(F) = \pi_{n+1}(Y, B)$.

We didn't need all this to take $(Y, B) = (K_N \# X, K_N \# A)$ and obtain a LHS — however, with $\pi_{\bullet+N}(K_N \# X/K_N \# A)$ in the definition of homology $h_\bullet(X, A)$ replaced by $\pi_{\bullet+N}(K_N \# X, K_N \# A)$. In this case it is the factorization axiom that needs explaining.

To this end, we refer to 22.2C and 23.4CD in [2], where it is proved (applying spectral sequences) that when Y and B are N-connected, the natural map $\pi_n(Y,B) \to \pi_n(Y/B)$ is an isomorphism for $n < 2N$ and epimorphism for $n = 2N$. From this, it can be derived that the replacement of one with the other in the definition of homology becomes inconsequential in the limit $N \to \infty$.

C. Two examples. (1) Let K_0 be any space and $K_N := \Sigma^N K_0$. This a Σ-spectrum, and up to homotopy equivalences any Σ-spectrum has this form. Then $h^\bullet(X;\mathcal{K}) = \varinjlim \pi(\Sigma^N X, \Sigma^{\bullet+N})$ are *stable cohomotopy groups* of X, while $h_\bullet(X;\mathcal{K}) = \varinjlim \pi_{\bullet+N}(\Sigma^N X^+)$ are *stable homotopy groups* of X which, due to generalized Freudenthal's suspension theorem (proved in 22.2C of [2] using spectral sequences) stabilize when $2N \geq \bullet$.

Taking $X = pt$ we find that the coefficients of the theory are stable homotopy groups of spheres $\pi_\bullet^{st}(S^0)$. Computing them is a separate branch of topology full of open problems. However, it is a theorem of Serre (proved using spectral sequences, see 26.3 in [2]) that all but $\pi_0^{st}(S^0) = \mathbb{Z}$ are finite. Consequently, when tensored with $\mathbb{Q}$, the (co)homology theory with coefficients in the spectrum $K_n = S^n$ turns into singular (co)homology with rational coefficients.

Every space X generates a Σ-spectrum $\Sigma^n X^+$, and maps $X \to Y$ induce morphisms $\Sigma^n X^+ \to \Sigma^n Y^+$ of the spectra. This suggests expanding the homotopy category of spaces to the category of spectra with morphisms $F : \mathcal{K}' \to \mathcal{K}$ defined as sequences of maps $F_n : K'_n \to K_n$ commuting (at least up to homotopy) with the structure maps of the spectra: $F_{n+1} \circ f'_n \sim f_n \circ \Sigma F_n$.

From this point of view, $h^n(X;\mathcal{K})$ is simply the set of homotopy classes of maps from the spectrum $\Sigma^N X^+$ to the shifted spectrum $\mathcal{K}_{+n} := \{K_{n+N}\}$, which generalizes to define cohomology of spectra:

$$h^\bullet(\mathcal{K}';\mathcal{K}) := \pi(\mathcal{K}', \mathcal{K}_{+\bullet}).$$

(2) Spaces $K_n := K(G,n)$ and the usual homotopy equivalences $K(G,n) \stackrel{g_n}{\to} \Omega K(G,n+1)$ define the *Eilenberg-MacLane* Ω-spectrum generating singular (co)homology with coefficients in G:

$$H^n(X;G) = \pi(X^+, K(G,n)), \quad H_n(X;G) = \varinjlim_{N\to\infty} \pi_{n+N}(K(G,N)\# X^+).$$

The first statement is familiar to us, but both follow from the abstract uniqueness argument outlined in Section A, because the dimension axiom (4) in this case is satisfied.

When G is a ring, or more generally: an abelian group homomorphism $G' \otimes G'' \to G$ is given, it defines a cohomology class in $H^{m+n}(K(G',m)\#K(G'',n);G)$ ($= Hom(G'\otimes G'', G)$ by Hurewicz' + Kunneth's theorems) and hence a homotopy unique map $K(G',m) \times K(G'',n) \to K(G; m+n)$. As we remarked in Section 24C, multiplications in cohomology are induced by such maps. From this example, one can guess how multiplicative structures of general extraordinary cohomology theories are encoded in terms of spectra.

In fact, any spectrum $\mathcal{K}$ can be replaced with an Ω-spectrum $\widehat{\mathcal{K}}$ by taking direct limits with respect to $g_n : K_n \to \Omega K_{n+1}$:

$$\widehat{K}_n := \varinjlim_{N\to\infty} \Omega^N K_{n+N} = \Omega \varinjlim_{N\to\infty} \Omega^N K_{n+N+1}.$$

That's why Ω-spectra are also called *infinite loop spaces.*

D. K-theory. In Lecture 15, we introduced the Grothendieck ring $K(X)$ of equivalence classes of virtual complex vector bundles over X, and proved in Section 15E that for *finite* CW-complexes, $K(X) = \pi(X^+, \mathbb{Z} \times BU)$ (recall that π here means π_b, hence X^+ rather than X), where $BU = \varinjlim BU_N \sim \mathbb{C}G(\infty,\infty)$ is the classifying space for the group $U = \varinjlim U_N$, the direct limit of block-diagonal inclusions $U_N \subset U_{N+1}$ of unitary groups.

`Lemma.` *U is weakly homotopy equivalent to ΩBU.*

`Proof.` From the fibered square of Hurewicz fibrations

$$\begin{tikzcd} E \arrow[r, "\Omega BG"] \arrow[d, "G"'] & EG \arrow[d, "G"] \\ E(BG) \arrow[r, "\Omega BG"] & BG \end{tikzcd}$$

where $EG \sim pt \sim E(BG)$ we see that both inclusions $G \hookrightarrow E$ and $\Omega BG \hookrightarrow E$ are weak homotopy equivalences — for any group G. $\square$

So, we can try to form an Ω-spectrum $\dots, \Omega^2 U, \Omega U, U, \mathbb{Z}\times BU$ and define $K^0(X) := \pi(X^+, \mathbb{Z}\times BU)$, $K^{-1}(X) := \pi(X^+, U)$, $K^{-2}(X) := \pi(X^+, \Omega U)$, and so on — except that our "spectrum" is infinite in the wrong direction. The deal is saved by celebrated

`Bott's periodicity theorem.` $\Omega U \sim \mathbb{Z} \times BU$.

It furnishes the 2-periodic Ω-spectrum $K_{2m} = \mathbb{Z} \times BU$, $K_{2m-1} = U$ which makes *complex K-theory*, defined for finite CW-complexes by $K^{2m}(X) := K(X)$, $K^{2m-1}(X) := \widetilde{K}(\Sigma X^+) = \pi(X^+, U)$, an extraordinary cohomology theory. The coefficient groups of complex K-theory are $K^{2m}(pt) = \mathbb{Z}$, $K^{2m-1}(pt) = 0$.

It is a multiplicative theory. The $\times$-product $K^0(X) \times K^0(\widetilde{X}) \to K^0(X \times \widetilde{X})$ is induced, as we already understand from Lecture 15, by tensor product of vector bundles. It extends to "odd $\times$ even $\to$ odd" and "odd $\times$ odd $\to$ even" degrees due to associativity of "#":

$$(\Sigma X^+)\#\widetilde{X}^+ = \Sigma(X \times \widetilde{X})^+ \text{ and } (\Sigma X^+)\#(\Sigma\widetilde{X}^+) = \Sigma^2(X \times \widetilde{X})^+.$$

We leave it as an exercise for the reader to figure out the expression of this in terms of the spectrum.

The Chern character defines a multiplicative morphism from complex K-theory to rational cohomology made 2-periodic:

$$\mathrm{ch} : K^0(X) \to H^{even}(X;\mathbb{Q}), \quad \mathrm{ch} : K^1(X) \to H^{odd}(X;\mathbb{Q})$$

which after tensoring $K^\bullet(X)$ with $\mathbb{Q}$ turns into an isomorphism. In fact this is a general property, at the bottom of which lies the fact that $\pi^{st}_\bullet(S^0) \otimes \mathbb{Q} = H_\bullet(pt;\mathbb{Q})$:

Any extraordinary cohomology theory h tensored with $\mathbb{Q}$ turns into singular cohomology with coefficients in $h^\bullet(pt)$.

Here is another general fact illustrated by the Chern character:

A morphism between two (co)homology theories is an isomorphism whenever it is an isomorphism of their coefficients.

Finally, real and quaternionic vector bundles lead to K-theories too (due to respective Bott periodicity theorems — this time with period 8), which turn out to be equivalent: $KO^\bullet = KSp^{\bullet+4}$. Their Ω-spectra (where all $\Omega K_n \sim K_{n-1}$) and coefficient groups are:

U/O	Sp/U	Sp	$\mathbb{Z} \times BSp$	U/Sp	O/U	O	$\mathbb{Z} \times BO$
0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

EXERCISES

237. Make sense of the spaces in the top row of the above table and verify the bottom one.

238. Verify exactness of the "dual Puppe sequence".

239. Explain why a (co)homology theory tensored with $\mathbb{Q}$ still satisfies the axioms (1-3).

240. Describe the $\times$-product in complex K-theory in terms of its spectrum.

241. Construct $\mathrm{ch} : K^1(X) \to H^{odd}(X;\mathbb{Q})$.

242. Show that $\mathbb{C}(\xi - 1)$, where ξ is the Möbius line bundle, has order 4 in $\widetilde{K}^0(\mathbb{R}P^4)$ (in fact it generates the group) which is therefore not isomorphic to $\widetilde{H}^{even}(\mathbb{R}P^4) \cong \mathbb{Z}_2^2$, although the coefficient groups of $K^{0/1}(-)$ and $H^{even/odd}(-)$ are isomorphic.

Lecture 29. Cobordisms

Here we will finally understand why Thom spaces and transversality theorems are named after the same person.

A. Bordism groups. By definition, a *singular n-fold* in a topological space X is a continuous map $\varphi : M \to X$ of a closed n-dimensional manifold M. It is called *bordant* to another singular n-fold $\widetilde{\varphi} : \widetilde{M} \to X$ if there exists an $(n+1)$-dimensional compact manifold W with boundary $\partial W = M \sqcup \widetilde{M}$ and a map $\Phi : W \to X$ such that $\Phi|_M = \varphi$ and $\Phi|_{\widetilde{M}} = \widetilde{\varphi}$.

The set of bordism equivalence classes is a group with respect to the operation of disjoint union $\varphi + \varphi' : M \sqcup M' \to X$, and in fact a $\mathbb{Z}_2$-vector space since $\varphi + \varphi$ is the boundary of the cylinder $\varphi \times \mathrm{id}_I : M \times I \to X$. This vector space is called the group of n-dimensional (unoriented) bordisms of X and is denoted $\Omega_n^O(X)$.

Maps $f : X \to Y$ induce homomorphisms $f_* : \Omega_n^O(X) \to \Omega_n^O(Y)$ via $f_\# : (M, \varphi) \mapsto (M, f \circ \varphi)$, and $f_0 \sim f_1 \;\Rightarrow\; f_{0*} = f_{1*}$ because $(M, f_0 \circ \varphi)$ and $(M, f_1 \circ \varphi)$ are bordant by $(f_t \circ \varphi) : M \times I \to X$.

`Theorem (the Pontryagin-Thom construction).`

$$\Omega_n^O(X) = \varinjlim_{N\to\infty} \pi_{n+N}(X^+ \# T\xi_N).$$

Here the *Thom spectrum* is made of the Thom spaces $T\xi_N$ of the tautological vector bundles ξ_N over $G(\infty, N)$, and of the maps $f_N : \Sigma T\xi_n = T(\xi_N \oplus \mathbb{R}) \to T\xi_{N+1}$ defined by standard inclusions $G(\infty, N) \subset G(\infty, N+1)$ which induce $\xi_N \oplus \mathbb{R}$ from ξ_{N+1} and thus induce maps between the bundle's Thom spaces.

`Proof.` We will indicate maps in both directions.

Given a singular n-fold $\varphi : M^n \to X$, smoothly embed M^n into S^{n+N} and denote by $(U, \partial U)$ a tubular neighborhood of M in the sphere. It is fibered over M with the fiber $(D^N, \partial D^N)$. Inducing the normal bundle of M in S^{n+N} from the universal bundle ξ_N over $BO_N = G(\infty, N)$, we may assume that the disk bundle π over M is induced from the universal disk bundle $(E, \partial E) \to BO_N$:

$$\begin{array}{ccc} (U,\partial U) & \xrightarrow{F} & (E,\partial E) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ M^n & \longrightarrow & BO_N \end{array}$$

Thus, we obtain a map $F : (U, \partial U) \to (E, \partial E)$. Together with $\varphi \circ \pi :$

$(U, \partial U) \to M \to X$, they define a map $(U, \partial U) \to (X \times E, X \times \partial E)$ and hence a map between the quotients:

$$U/\partial U \to (X \times E)/(X \times \partial E) = X^+ \# T\xi_N.$$

Pre-composing it by the quotient map $S^{n+N} \to S^{n+N}/(S^{n+N} - \mathring{U}) = U/\partial U$, we obtain an $(n+N)$-spheroid $\widehat{\varphi} : S^{n+N} \to X^+ \# T\xi_N$.

The construction can be repeated for a bordism $\Phi : W^{n+1} \to X$ between $\varphi_0 : M_0 \to X$ and $\varphi_1 : M_1 \to X$ by embedding W^{n+1} into $S^{n+N} \times I$ so that its two boundaries M_0 and M_1 are embedded into $S^{n+N} \times 0$ and $S^{n+N} \times 1$ respectively (Figure 65). Inducing the normal bundle of W from ξ_N and proceeding as before we obtain a homotopy $\widehat{\Phi} : S^{n+N} \times I \to X^+ \# T\xi_N$ between $\widehat{\varphi}_0$ and $\widehat{\varphi}_1$.

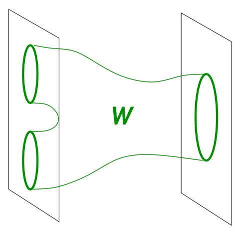

Figure 65: Bordism in $S^{n+N} \times I$

In the reverse direction, given a spheroid $\psi : S^{n+N} \to X^+ \# T\xi_N$, on the inverse image $\mathcal{U}$ of the open set $X \times (T\xi_N - \infty)$, we have two maps defined by composing $\psi|_{\mathcal{U}}$ with the projections to the factors: $\pi : \mathcal{U} \to X$ and $F : \mathcal{U} \to (T\xi_N - \infty)$ (= the total space of the tautological bundle ξ_N over $G(\infty, N)$).

The inverse image by F^{-1} of the zero section of ξ_N and of a closed disk neighborhood V of it is closed (and hence compact) in S^{n+N}, and we conclude (by the weak topology of $T\xi_N$) that $\mathcal{V} := F^{-1}(V) \subset \mathcal{U} \subset S^{n+N}$ is mapped by F to a finite skeleton and lands, therefore, in the Thom space $T\xi$ of the tautological bundle over a finite dimensional grassmannian $G(K+N, N)$.

Moreover, by smooth approximation and the elementary transversality theorem we may assume that F is smooth on $\mathcal{V}$ and is transverse to the zero section of ξ. The inverse image $F^{-1}(G(K+N, N))$ of the zero section is a closed submanifold $M \subset \mathcal{V}$ of codimension N. The projection $\varphi := \pi|_M : M \to X$ is a singular n-dimensional manifold in X. It is smoothly embedded into S^{n+N}, and it is clear that using this embedding in the direct construction, we will end up with the $(n+N)$ spheroid $\widehat{\varphi}$ in $X^+ \# T\xi_N$ homotopic to the given ψ.

Finally, applying this procedure to a homotopy $(\psi_t) : S^{n+N} \times I \to X^+ \# T\xi_N$ we end up with a manifold $W \in S^{n+N} \times I$ with boundary $\partial W \subset S^{n+N} \times \partial I$ and a bordism $\Phi : W \to X$ between the singular n-folds φ_0 and φ_1 corresponding to ψ_0 and ψ_1.

B. Cobordism rings. Once we have a spectrum, we have the corresponding *co*homology theory (at least for finite CW-complexes X), the *cobordism theory* in our case:

$$\Omega_O^\bullet(X) := \varinjlim_{N\to\infty} \pi(\Sigma^N X^+, T\xi_{\bullet+N}).$$

It comes with a graded ring structure. Namely the direct sum $\xi_k \times \xi_l$ of the universal bundles over $BO_k \times BO_l$ is induced from ξ_{k+l} over BO_{k+l} by a fiberwise bundle map which in its turn induces a map of Thom spaces:

$$T\xi_k \# T\xi_l = T(\xi_k \times \xi_l) \to T\xi_{k+l}.$$

By the general machinery of spectra this defines the cross-product $\Omega_O^k(X) \times \Omega_O^l(\widetilde{X}) \to \Omega_O^{k+l}(X \times \widetilde{X})$, and by means of the diagonal map $X \to X \times X$ (when $\widetilde{X} = X$) the cobordic cup-product on $\Omega_O^\bullet(X)$.

Yet, a cohomology theory defined by means of its spectrum remains a tautological homotopy-theoretic study of the spectrum unless the theory has another, more geometric interpretation. In the case of cobordism theory, such an interpretation becomes apparent when the space X itself is a closed manifold of certain dimension m, because in this case we have the *cobordic Poincaré isomorphism*:

`Proposition.` $\Omega_\bullet^O(X^m) = \Omega_O^{m-\bullet}(X^m)$.

`Proof.` This is a variant of the Pontryagin-Thom construction. Given a singular n-fold $\varphi : M^n \to X^m$, which by virtue of approximation can be assumed smooth, we can smoothly embed M^n into $\mathbb{R}^N$ and induce the tubular neighborhood U of $M^n \subset \mathbb{R}^N \times X^m$ from the universal D^{N+m-n}-bundle over BO_{N+m-n}. Contracting the complement of U we obtain a map

$$\widetilde{\varphi} : \Sigma^N X^+ \to \Sigma^N X^+ / (\Sigma^N X^+ - \mathring{U}) = U/\partial U \to T\xi_{N+n-m}$$

representing the Poincaré-dual cobordism class.

In the reverse direction, $\psi : \Sigma^N X^+ \to T\xi_{N+m-n}$ is smoothened inside $\mathbb{R}^N \times X^m$ and made transverse to the zero section. Then $M^n := \psi^{-1}(\textit{zero section})$ is a closed submanifold in $X \times \mathbb{R}^N$ whose projection to X defines the singular n-fold φ such that $\widetilde{\varphi} \sim \psi$. $\square$

Thus, a cobordism class $\alpha \in \Omega^n_O(X^m)$ is represented by a singular manifold $\varphi : M \to X$ of *co*dimension n. Given a (smooth) map $f : Y \to X$ of closed manifolds, the class $f^*\alpha$ is Poincaré-dual to a singular manifold $N \to Y$ of the same codimension. It is obtained — in the spirit of intersection theory — by smoothly approximating and perturbing $\varphi \times f : M \times Y \to X \times X$ to make it transverse to the diagonal $\Delta \subset X \times X$, and projecting the inverse image N of the diagonal from $M \times Y$ to Y.

Given two cobordism classes Poincaré-dual to singular manifolds $\varphi : M \to X$ and $\widetilde{\varphi} : \widetilde{M} \to \widetilde{X}$, their cobordic cross-product is Poincaré-dual to the product map $\varphi \times \widetilde{\varphi} : M \times \widetilde{M} \to X \times \widetilde{X}$. Consequently, the Poincaré-dual expression of cobordic cup-product is obtained (when $\widetilde{X} = X$) by perturbing $\varphi \times \widetilde{\varphi}$ to make it transverse to $\Delta \subset X \times X$, and mapping the inverse image of Δ back to $\Delta = X$.

In the special case of $X = pt$ we obtain the theory's coefficient ring $\Omega^O_\bullet = \Omega^{-\bullet}_O$ known as the *Thom ring* of unoriented (co)bordisms. It is the graded $\mathbb{Z}_2$-algebra of equivalence classes of closed manifolds with respect to the bordism equivalence relation, with the operations of disjoint union and Cartesian product in the roles of addition and multiplication, and with grading $\bullet$ defined by the (negative, in the *co*bordism interpretation) dimension of the manifolds.

C. Other cobordism theories. There are many of them, distinguished by the choice of an additional structure the stable tangent bundles of singular manifolds $M \to X$ and bordisms between $W \to X$ them are required to carry.

Requiring that M and W are oriented, and ∂W is equipped with the orientation induced by that of W (the exterior normal vector followed by a right-oriented basis for ∂W is a right-oriented basis for W), we obtain the theories $\Omega^{SO}_\bullet$ and $\Omega^\bullet_{SO}$ of oriented (co)bordisms. Here the opposite of a singular manifold $\varphi : M \to X$ is obtained by reversing the orientation of M: Together, they bound the cylinder $\varphi \times \mathrm{id}_I : M \times I \to X$. The spectrum of these theories consists of the Thom spaces $T\xi^{SO}_N$ of universal vector bundles over BSO_N.

Requiring that M and W are *stably almost complex*, i.e. their tangent bundles, possibly after adding a trivial bundle, are equipped with complex structures (for the boundary ∂W — compatible with that of W in the sense that we let the reader to clarify), we arrive at the theories $\Omega^U_\bullet$ and $\Omega^\bullet_U$ of complex (co)bordisms. Note that a stably almost complex structure of M includes an orientation defined by the complex orientation of the stabilized tangent bundle and by the standard orientation of the trivial "stabilizing" bundle. The notion of the opposite to a singular bordism is a bit tricky here. If $\tau_M \oplus \mathbb{R}^k$ is made a complex vector bundle, then $\tau_M \oplus \mathbb{R}^K \oplus \mathbb{C}$ is stably equivalent to it. The opposite one is obtained by replacing $\mathbb{C}$ with $\overline{\mathbb{C}}$ — the trivial bundle $\mathbb{R}^2$ equipped with the conjugate complex structure.

The Thom spaces $T\xi_N^U$ of the universal complex vector bundles over BU_N are related by maps $\Sigma^2 T\xi_N^U \to T\xi_{N+1}^U$ obtained by inducing $\xi_N^U \oplus \mathbb{C}$ from ξ_{N+1}^U. To make of the sequence $T\xi_N^U$ the Thom spectrum of the complex (co)bordism theory, one needs to interlace it with the sequence of suspensions $\Sigma T\xi_N^U = T(\xi_N^U \oplus \mathbb{R})$.

Likewise, one can introduce symplectic (co)bordism theories $\Omega_\bullet^{Sp}$ and $\Omega_{Sp}^\bullet$ by imposing stably quaternionic structures on the tangent bundles of manifolds, with the Thom spectrum

$$\ldots,\ T\xi_N^{Sp},\ \Sigma T\xi_N^{Sp},\ \Sigma^2 T\xi_N^{Sp},\ \Sigma^3 T\xi_N^{Sp},\ T\xi_{N+1}^{Sp},\ \ldots,$$

where ξ_N^{Sp} is the universal quaternionic vector bundle over BSp_N.

We stop here, but there are many other meaningful (co)bordism theories and interesting Thom spectra.

D. Thom's theorem. This is a celebrated 1954 result describing the coefficient rings of the four aforementioned cobordism theories, i.e. the rings of bordism classes of closed manifolds whose tangent bundles are equipped with stably complex or stably quaternionic structures, or merely orientations, or no structures at all, with the operations of disjoint union and Cartesian product. In effect, it computes stable homotopy groups (their ranks, in the SO-case) of the corresponding Thom spectra.

`Theorem.` *The rings of complex and quaternionic bordisms are free polynomial algebras over $\mathbb{Z}$ with generators of degrees $2k$ (resp. $4k$) for which, over $\mathbb{Q}$, projective spaces $\mathbb{C}P^k$ and $\mathbb{H}P^k$ can be taken:*

$$\Omega_\bullet^U(pt) \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, \ldots],$$
$$\Omega_\bullet^{Sp}(pt) \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{H}P^1, \mathbb{H}P^2, \mathbb{H}P^3, \ldots].$$

The ring of oriented bordisms tensored with $\mathbb{Q}$ is a free polynomial algebra on generators $\mathbb{C}P^{2k}$ of degree $4k$:

$$\Omega_\bullet^{SO}(pt) \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \ldots].$$

The ring of unoriented bordisms is a free polynomial $\mathbb{Z}_2$-algebra on generators x_k of degree k such that $k+1$ is not a power of 2:

$$\Omega_\bullet^O(pt) = \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, \ldots],$$

where for the generators x_{2k}, projective spaces $\mathbb{R}P^{2k}$ can be taken.

In all four theories, the bordism class of a manifold is uniquely determined (in the SO-case — modulo torsion) by the characteristic numbers of the manifold (Chern, Pontryagin, Pontryagin, and Stiefel-Whitney respectively).

`Examples.` (1) Every non-orientable surface contains a Möbius band, in which the $\mathbb{Z}_2$-valued self-intersection of the middle circle equals 1. This implies that for $M = P^2_g$ or K^2_g, $w_1^2(\tau_M) \neq 0$, and hence $\langle M, w_1^2(\tau_M)\rangle = 1$. Thus, M is bordant to $\mathbb{R}P^2$.

(2) Every closed 4-dimensional manifold is (unorientably) bordant to $\mathbb{R}P^4, (\mathbb{R}P^2)^2$, their disjoint union, or $\emptyset$. E.g. $\mathbb{C}P^2$ is bordant to $\mathbb{R}P^2 \times \mathbb{R}P^2$, as can be found by computing Stiefel-Whitney numbers of both. The total Chern class of $\tau_{\mathbb{C}P^2}$ is $(1+x)^3 \equiv 1+3x+3x^2 \mod x^3$ (where $x = c_1(L^*)$), meaning that $\tau_{\mathbb{C}P^2}$ has $w_2 = \rho_2 x$, $w_4 = \rho_2 x^2$ and consequently $\langle [\mathbb{C}P^2], w_2^2\rangle = \langle [\mathbb{C}P^2], w_4\rangle = 1$. We leave computing the Stiefel-Whitney numbers of $\mathbb{R}P^2 \times \mathbb{R}P^2$ as an exercise.

(3) According to a general theorem quoted in Section 28D, $\Omega^\bullet_{SO}(X) \otimes \mathbb{Q} \cong H^\bullet(X; \Omega^\bullet_{SO}(pt) \otimes \mathbb{Q})$. Consequently, any rational cohomology class of a closed oriented manifold X is Poincaré-dual to (a rational multiple of) the fundamental class $\varphi_*[M]$ of a "singular" closed oriented manifold $\varphi : M \to X$, where (thanks to Thom's theorem) M can be taken as a disjoint union of products of $\mathbb{C}P^{2k}$.

E. Signature. The signature $\sigma(M)$ of the intersection form on the middle homology $H_{2m}(M)$ of a closed oriented $4m$-dimensional manifold turns out to be bordism-invariant. The proof can be obtained form the Poincaré isomorphism between two LESequences:

$$\begin{array}{ccccc} H^{2m}(W;\mathbb{Q}) & \xrightarrow{i^*} & H^{2m}(\partial W;\mathbb{Q}) & \xrightarrow{\delta^*} & H^{2m+1}(W,\partial W;\mathbb{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ H_{2m+1}(W,\partial W;\mathbb{Q}) & \xrightarrow{\partial_*} & H_{2m}(\partial W;\mathbb{Q}) & \xrightarrow{i_*} & H_{2m}(W;\mathbb{Q}) \end{array}$$

It shows that i_* is adjoint to ∂_*, and that $\ker i_*$ is an isotropic subspace in $H_{2m}(\partial W;\mathbb{Q})$ of half its dimension. It cannot exist unless the non-degenerate symmetric bilinear form has signature 0.

The *Hirzebruch signature formula* below is derived from Thom's description of $\Omega^{SO}_\bullet(pt) \otimes \mathbb{Q}$. Express homogeneous terms L_m of

$$\prod_i x_i \frac{e^{x_i} + e^{-x_i}}{e^{x_i} - e^{-xi}} = 1 + L_1(x^2) + \cdots + L_m(x^2) + \cdots$$

via elementary symmetric functions as $\hat{L}_m(\sigma_1(x^2), \sigma_2(x^2), \dots)$. Then

$$\sigma(M) = \langle [M], \hat{L}_m(p_1(\tau_M), p_2(\tau_M), \dots)\rangle.$$

F. Cohomological operations. We have developed the intuition that cohomology of interesting spaces might very well be computable, but finding all homotopy groups, even of such simple spaces as spheres, is a monstrous task. How does one succeed in computing all stable homotopy groups of Thom spectra, especially of $T\xi_N^O$, where the answer is far from trivial?

Here is the starting idea. Explore cohomology of $X := T\xi_N$ and of Eilenberg-MacLane spaces well enough in order to find a map $X \to Y = \prod_i K(G_i, n_i)$ which would establish an isomorphism in cohomology, at least in a range of degrees growing as $N \to \infty$. Then, applying Whitehead's homological theorem, conclude that the map induces an isomorphism of homotopy groups in that growing range, and enjoy the fact that all homotopy groups of Y are known.

Next, homotopically a map $f : X \to K(G, n)$ is an element $f^*[F_{G,n}] \in H^n(X; G)$. But it comes not alone: Every element $\alpha \in H^{n+k}(K(G, n))$ is mapped to some element $f^*\alpha \in H^{n+k}(X; G)$. In other words, α corresponds to a map $\varphi_\alpha : K(G, n) \to K(G, n + k)$, and $f^*\alpha = (f \circ \varphi_\alpha)^*[F_{G,k+n}]$.

More generally, elements of $\pi(K(G, n), K(G', n'))$ (which merely represent cohomology classes in $H^{n'}(K(G, n); G'))$ define natural *cohomological operations* $H^n(X; G) \to H^{n'}(X; G')$ working coherently for all X. Even more generally, cohomological operations can be defined in all extraordinary cohomology theories and between different cohomology theories, and correspond to homotopy classes of maps between their spectra. Yet, the most relevant ones for computing $\Omega^O_\bullet(pt)$ are those which operate on $\mathbb{Z}_2$-cohomology and are *stable*, meaning that they commute with suspensions. As a set they form

$$\mathcal{A}_2 := \varprojlim_{N\to\infty} \widetilde{H}^{\bullet+N}(K(\mathbb{Z}_2, N); \mathbb{Z}_2)$$

with respect to homomorphisms induced by the structure maps $f_N : \Sigma K(\mathbb{Z}_2, N) \to K(\mathbb{Z}_2, N + 1)$ of the Eilenberg-MacLane spectrum, combined with suspension isomorphisms. Composition of cohomological operations make it an associative *Steenrod algebra*, and $\mathbb{Z}_2$-cohomology of any space is a module over it.

The structure of $\mathcal{A}_2$ is most efficiently described by its action on $\widetilde{H}^\bullet(T(\xi_1 \times \cdots \times \xi_1); \mathbb{Z}_2)$. The description carries over to $\widetilde{H}^\bullet(T\xi_N; \mathbb{Z}_2)$ by the splitting principle. It turns out that in the limit $N \to \infty$ the latter becomes a free $\mathcal{A}_2$-module with generators corresponding to monomials in $\mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, \ldots]$, and this determines all stable homotopy groups of the Thom spectrum.

EXERCISES

243. Show that $T(\eta \times \zeta) = T\eta \# T\zeta$.

244. Derive the intersection theory descriptions of the cobordic pull-back f^* and cross-product from their spectrum-theoretic definitions.

245. Show that S^2 equipped with the following two stably almost complex structures are not bordant as stably almost complex manifolds: One is the complex structure of $\mathbb{C}P^1$, the other is obtained by adding to τ_{S^2} the trivial normal bundle ν_{S^2} of S^2 in $\mathbb{R}^3$, then adding another trivial line $\mathbb{R}^1$, and then identifying $\tau_{S^2} \oplus \nu_{S^2} \oplus \mathbb{R}^1$ with the trivial bundle $\mathbb{R}^4 = \mathbb{C}^2$.

246. Show geometrically that P_g^2 and K_g^2 are bordant to $\mathbb{R}P^2$.

247. Compute Stiefel-Whitney numbers of $\mathbb{R}P^4$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$.

248. Show that $\hat{L}_1 = p_1/3$, $\hat{L}_2 = (7p_2 - p_1^2)/45$, and verify the Hirzebruch signature formula for $\mathbb{C}P^2$, $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$.

249. Compute Chern characteristic numbers of $\mathbb{C}P^3$, $\mathbb{C}P^2 \times \mathbb{C}P^1$, and $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, and compare them with those of the flag manifold $F_3(\mathbb{C})$ to express the complex bordism class of the latter as a rational linear combination of the former.

250. Let $f : X^m \to Y^n$ be a smooth map between closed oriented manifolds. Pick a smooth embedding $\varphi : X \to \mathbb{R}^N$ and denote by U a tubular neighborhood of X embedded in $Y \times \mathbb{R}^N$ by $f \times \varphi$. Show that the compositions of two Thom isomorphisms with the natural factorization map (the middle arrow)

$$H^\bullet(X) \stackrel{\cong}{\leftarrow} H^{\bullet+N+n-m}(U, U - X) \to H^{\bullet+N+n-m}(X \times \mathbb{R}^N, X \times (\mathbb{R}^N - 0)) \stackrel{\cong}{\to} H^{\bullet+n-m}(Y)$$

coincides with the composition $\alpha \mapsto D_Y^{-1} f_* D_X \alpha$ of two Poincaré isomorphisms with the natural map f_* in homology.

`Remark.` This "wrong direction" operation (somewhat in the spirit of the Pontryagin-Thom construction in Proposition from Section B) of taking *direct image* of α is known as cohomological *push-forward* $f_!$ and has a meaning of "integration over the fibers" of the map f. The reversely defined homological *pull-back* $f^! : H_\bullet(Y) \to H_{\bullet+m-n}(X)$ is interpreted as taking inverse images of cycles.

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