Geometry of Surfaces and the Gauss–Bonnet Theorem

1. Surfaces.

| Plane | Cylinder | Sphere | Mask |
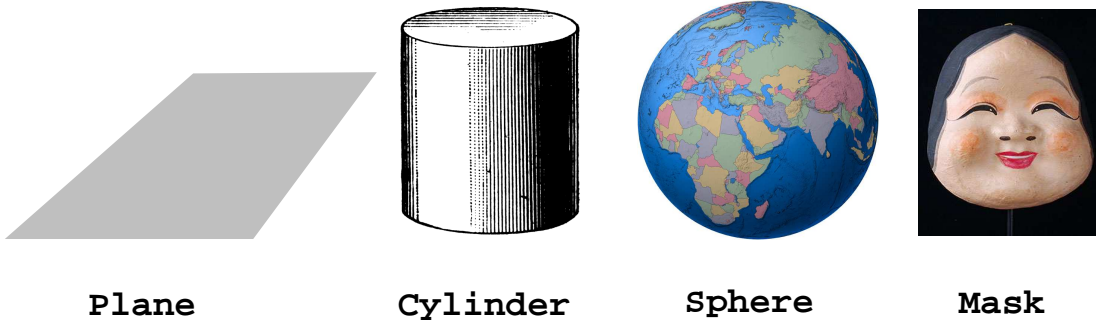
2. Topology vs. Geometry.

Which of the shown surfaces can be *deformed* into each other (i.e. transformed into each other without cutting or pasting) and which cannot?

Those that can are said to have **the same topology**.

Make cuts in a sheet of paper to obtain a surface having the same topology as the mask.

The resulting sheet of paper and the mask will still have different **geometry**. Indeed, in geometry, exact shapes and sizes matter. Deforming (such as stretching) changes shapes and sizes.

3. Bent or Curved?

A sheet of paper can be **bent** in space to form a new shape. The following experiment illustrates the fact that bending does not change *intrinsic* geometry of the surface (i.e. distances and angles as they would be measured by creatures that cannot leave the surface).

Wrap a cylinder with a sheet of paper, stretch a shoelace as shown, and trace it with marker. Then unwrap the paper — the trace will be a straight line.
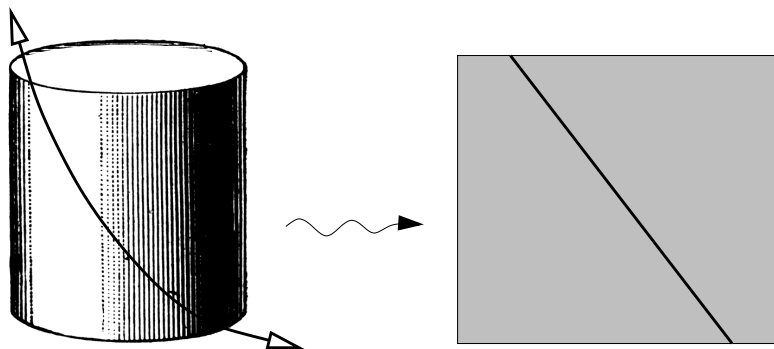
Find a solid object that has the shape of a cone and try the same experiment with it. What do you find?

Try to wrap a hemisphere with a sheet of paper — and fail! This is because the sphere is **curved**.

Curvature is what prevents one surface being wrapped around another without stretching or wrinkling. It is an *intrinsic* geometric property of a surface and does not disappear under bending.

How to *quantify* curvature?

`Question:` Are spheres of different radii curved differently?

Try (or imagine!) cutting out pieces of two different size rubber balls and wrapping one about another without stretching of wrinkling.

4. The Method. The following picture illustrates the main idea: A smooth surface can be *approximated* with a **polyhedron**, i.e. a surface formed by flat polygons connected at their sides and vertices.
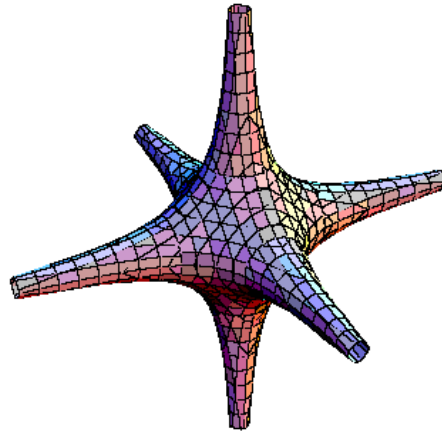
Visit *http://www.vismath.de/vgp/content/surfaces/* and play with the applet *Sphere* to see how polyhedra with more and more faces (which become smaller and smaller) provide better and better approximations to the surface and make it look smoother and smoother.

All geometric properties of surfaces should somehow be seen in geometric properties of polyhedra.

5. Curvature Resides at Vertices. From any polyhedron, cut out a piece of the surface that contains no vertices. Then it is possible to unbend the piece along edges to make it lie flat on the plane.
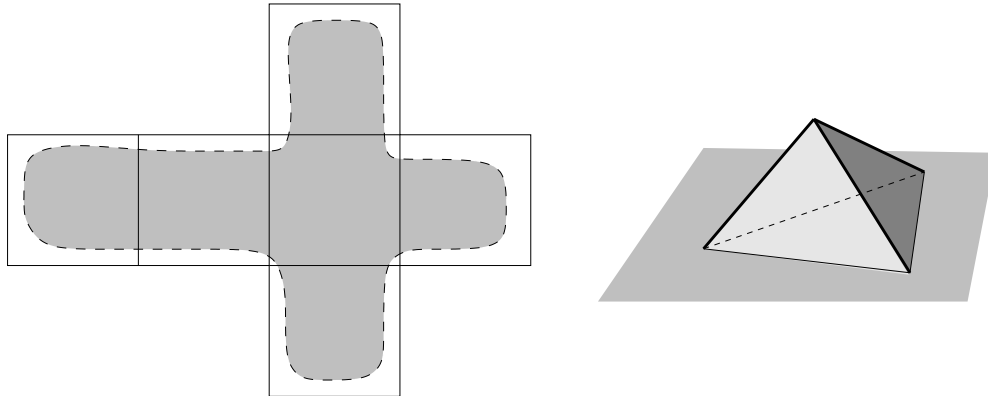
In the contrary, even a small piece around a vertex will hardly lie flat on the desk (unless cut or folded).

6. The Angular Defect. To be made of a flat piece of paper, the corner near a vertex of a polyhedron must have the adjacent angles adding up to the full angle, i.e. 360°.

The difference

$$360° - \text{ sum of angles at a vertex}$$

is called the ***angular defect of the vertex***.

The angular defect of a vertex is the measure of the amount of curvature residing at this vertex.

zero angular defect

positive angular defect

negative angular defect
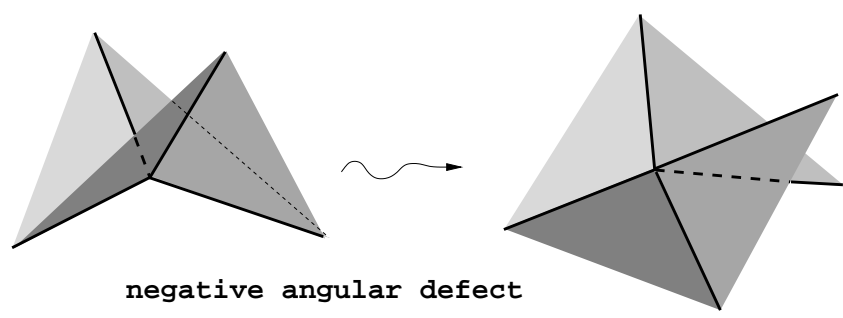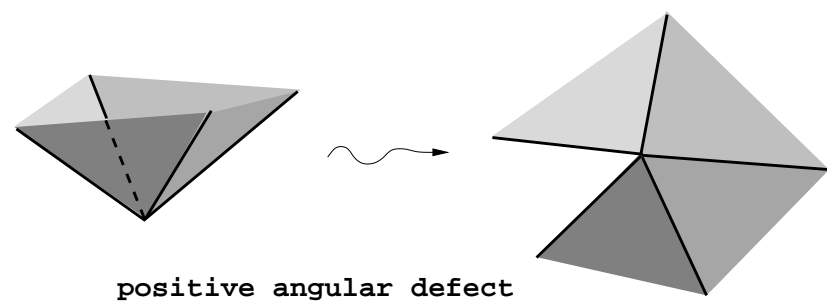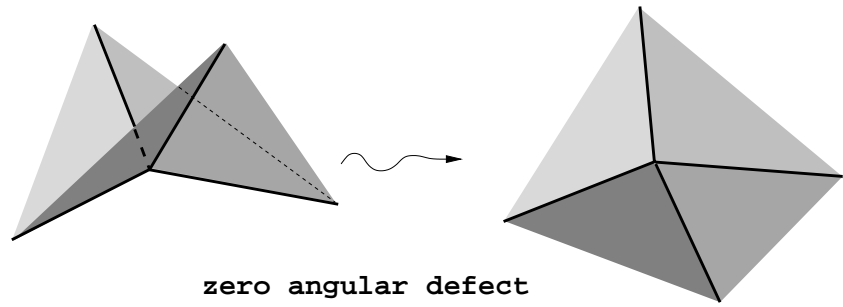
Control question: When the bird-like corner at the left bottom picture flaps its wings, does the angular defect change?

Brain-teaser: Gift bags have four corners at the bottom, but are sold folded flat. Examine such a bag and explain how this is done.

7. Total Angular Defects. Fill in the table.

(Semi)regular polyhedron	Angular defect of each vertex	Number of vertices	Total angular defect
Triangular prizm	$360° - 90° - 90° - 60°$ $= 360° - 240°$ $= 120°$	6	$120° \times 6 = 720°$
Tetrahedron			
Octahedron			
Cube			
Icosahedron			
Dodecahedron			
Soccer ball			

8. Why is the total defect the same? Let
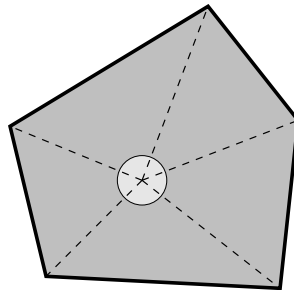V, E, and F denote the numbers of vertices, edges, and faces of a given polyhedron. By definition,

Defect of one vertex $= 360° -$ Sum of angles at this vertex.

Summing up we find that

Total defect of all vertices $=$

$$360° \times V - \text{ Sum of all angles at all vertices.}$$

Remember this. Now change the point of view. The sum of all angles at all vertices is the same as the sum of all angles in all faces.

From the picture we see that

Sum of angles in one face $=$
$180° \times$ Number of sides of this face $- 360°$

Summing up we find that

Sum of angles of all faces $=$
$180° \times$ Total number of sides of all faces $- 360° \times F$.

Taking in account that each edge is a side to exactly two faces, we can rewrite this result this way:

Sum of all angles at all vertices $= 360° \times E - 360° \times F$.

Now combine this with what was asked to remember.

Theorem. *Total angular defect $= 360° \times (V - E + F)$.*

9. Geometry vs. Combinatorics. Let us call the total angular defect of a polyhedron P divided by $360°$ the **_Gauss number_** of P, and $V - E + F$ the **_Euler number_** of P. The Gauss number characterizes geometry of P while the Euler number characterizes the *combinatorics* of P. The theorem says that for every polyhedron P,

*The Gauss Number of P = The Euler Number of P.*

Thus combinatorics of a polyhedron puts constraints on geometry of *this* polyhedron, and conversely, geometry of a polyhedron puts constraints on combinatorics of it. This relation between geometry and combinatorics is remarkable but not surprising. Now we will deduce from it that, *given any two polyhedra, P and T, The Gauss Number of P = The Euler Number of T, if only P and T have the same topology.*

For this, we will construct two polyhedra, P' and T', which have the same combinatorics (and possibly different geometry), and such that P' has the same geometry as P (and possibly different combinatorics), and T' the same geometry as T (and possibly different combinatorics).

The idea is very simple. Partitioning faces of a given polyhedron into smaller faces changes combinatorics of it, but does not change geometry. Therefore the Gauss

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number does not change, and by the theorem the Euler number remains the same too.
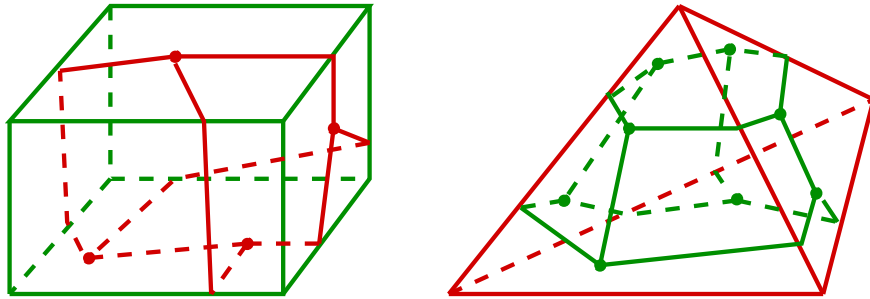
The idea is illustrated here in the example when P is a rectangular box, and T is a tetrahedron. Since P and T have the same topology, we can draw a picture of T on the surface of P (see the left figure, where thick dots show vertices of T on the surface of P). The edges of P and the lines connecting the thick dots form edges of a new polyhedron which we call P'. It has the same geometry as P and in particular the same total angular defect, because all newly created vertices have zero angular defects.

Exercise. On the left figure, locate 10 vertices of P' which are not those of P and show that each of them has zero angular defect.

Now deforming P into T, we obtain the picture of P drawn on the surface of T (the right figure). As in the first case, all the lines partition faces of T into faces of the new polyhedron, T', which has the same combinatorics as P' but the same total angular defect as T.

Exercise. Find directly the numbers V, E, F for P' (on the left figure) and T' (on the right figure) and check that they coincide.

We conclude that (explain each equality!)

The Euler Number of P = The Gauss Number of P =
The Gauss Number of P' = The Euler Number of P' =
The Euler number of T' = The Gauss Number of T' =
The Gauss Number of T = The Euler Number of T.

We have proved

The Gauss-Bonnet Theorem for Polyhedra.
The Gauss and Euler numbers of every polyhedron are equal to each other and depend only on the topology of the polyhedron.

Corollary. *If a polyhedron has the same topology as the sphere, then its Euler number $V - E + F = 2$, and the total sum of its angular defects is* $720°$.

Brain-teaser. Compute the total sum of angular defects of the polyhedron that has the same topology as a torus, and compare the result with the Euler number $V - E + F$.

Exercise. Find the Gauss and Euler numbers of every polyhedron that has the same topology as the surface of a pretzel.
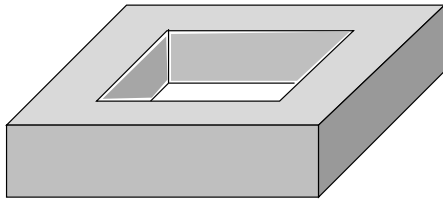
Torus
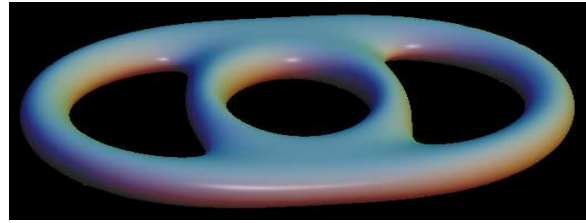
Pretzel

10. Gaussian Curvature. We now return to geometry of smooth surfaces. Such surfaces can be approximated by polyhedra with lots of tiny faces and lots of vertices with very small angular defects. The sum of angular defects of those vertices that lie in a region of the surface characterizes curvature of this region. It is called the *__Gaussian curvature__* of the region. If non-zero, this region of the surface, however small, cannot be unbent into a piece of plane or wrapped by a piece of paper without distortion or wrinkling.
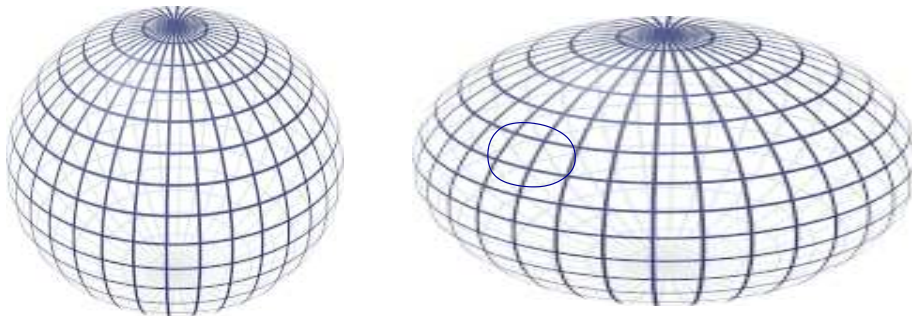
By choosing better and better approximations of a surface and applying the Gauss–Bonnet Theorem for Polyhedra, we obtain the following result.

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The Gauss–Bonnet Theorem for Surfaces.

The total Gaussian curvature of a closed surface depends only on the topology of the surface and is equal to 2π times the Euler number of this surface.

The factor 2π (instead of 360°) occurs here because Gauss measured the full angle not by 360° but by the ratio, equal to 2π, of the circumference around the vertex of the full angle to the radius of the circle.

Problem. Show that spheres have constant Gaussian curvature (i.e. the same near all points of the sphere) and find the value of the Gaussian curvature of the sphere of redius R.

Historical remarks. The Gauss–Bonnet Theorem for polyhedra that have the same topology as the sphere was actually discovered by René Descartes (1596–1650). Carl Friedrich Gauss (1777–1855) introduced the notion of curvature, proved that it does not change under bending of surfaces, and established what we call the Gauss–Bonnet Theorem for Surfaces. Pierre Ossian Bonnet (1819–1892) found a generalization of this theorem that remains valid for surfaces which are not necessarily *closed* (like sphere, torus, or pretzel) but instead may have the same topology as a disk, and may have boundary.