EXERCISES IN SYMPLECTIC GEOMETRY [1]

1. Prove the Linear "Relative Darboux Theorem" (page 6).
2. Verify that the graph of a linear map $A : X \to X^*$ is Lagrangian in $V = X^* \oplus X$ if and only if A is symmetric: $A^* = A$.
3. Write explicitly the Plücker relation $\phi \wedge \phi = 0$ for an exterior 2-form $\phi = \sum_{1 \leq i < j \leq 4} \phi_{ij} x_i \wedge x_j$ in $\mathbb{R}^4$ and show that it is a non-degenerate quadratic equation of signature $(+++---)$. Derive that the grassmannian $Gr_{2.4}(\mathbb{R})$ is diffeomorphic to the quotient of $S^2 \times S^2$ by the simultaneous antipodal map $(x, y) \mapsto (-x, -y)$, and give the analogous description of the Lagrange grassmannian Λ_2.
4. Verify the formula (on page 8) for the Poisson bracket of quadratic Hamiltonians in Darboux coordinates.
5. Show that the adjoint action of the symplectic group on its Lie algebra coincides with the its action by the changes of variables in the quadratic hamiltonians. Derive from this that the center of $Sp(2n, \mathbb{R})$ consists of two elements $\pm E_{2n}$.
6. Prove that the one-parametric subgroup in $Sp(2n, \mathbb{R})$ generated by a positive definite quadratic Hamiltonian is pre-compact.
7. For $\omega = \sum \lambda_l p_k \wedge q_k$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$, show that the ω-area of the unit disk on any oriented plane L (with respect to the Euclidean norm $\sum (p_k^2 + q_k^2)$ in the ambient space) does not exceed $\pi \lambda_1$. (*Hint:* First show this this for $\lambda_1 = ... = \lambda_n = 1$ using the Cauchy–Schwarz inequality for the Hermitian form $\langle z, w \rangle = \sum z_k \bar{w}_k$.)
8. Show that an infinitesimal symplectic transformation $H \in sp(V)$ is similar to $-H$ (and therefore its eigenvalue spectrum and the whole Jordan block structure is invariant under the transformation $\lambda \mapsto -\lambda$).
9. Show that the symmetric operator S in the polar factorization $A = SU$ of a linear symplectic transformation A, in a suitable decomposition $\oplus_{k=1}^n \mathbb{R}^2$ of the symplectic space into the direct sum of n orthogonal (with respect to both the symplectic and Euclidean structures) symplectic planes, is described as the superposition of symplectic hyperbolic rotations in these planes (or, equivalently, as the time-1 map in the flow of the Hamiltonain $\sum_{k=1}^n \lambda_k p_k q_k$ with suitable real λ_k).
10. Linearize the pendulum equation near the upper (unstable) equilibrium, sketch the phase portrait, and describe the phase flow in terms of hamiltonian hyperbolic rotations.
11. Describe (and sketch) the partition of the 3-dimensional space of quadratic forms on the symplectic plane $(\mathbb{R}^2, p \wedge q)$ into the orbits of the adjoint

[1] Math 242, Fall'21, A. Givental, references to "Symplectic geometry" by Arnold–Givental in vol. 4 of Springer's EMS series.

action of $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$. Indicate in your sketch the region of sign-definite quadratic hamiltonians.

12. Show that the $\mathbb{H}$-valued positive-definite sesquilinear form in $\mathbb{H}^n$ (with *right* multiplication by quaternionic scalars) which is given by the formula $\langle x, y\rangle := \sum_{m=1}^{n} x_m^* y_m$ has the structure $Z(x,y) + jW(x,y)$ where Z and W are respectively Hermitian and complex symplectic forms on $\mathbb{H}^n = \mathbb{C}^{2n}$ with respect to the complex structure defined by i. Express Z via W (and j) and *vice versa* W via Z (and j). Derive that $GL_n(\mathbb{H}) \cap U_{2n}$ (where $U_{2n} = Aut_{\mathbb{C}}(Z)$) coincides with $GL_n(\mathbb{H}) \cap Sp(2n,\mathbb{C})$ (where $Sp(2n,\mathbb{C}) = Aut_{\mathbb{C}}(W)$ and therefore coincides with the compact groups Sp_n of quaternionic-linear automorphismsm of the $\mathbb{H}$-valued sesquilinear form $\langle\cdot,\cdot\rangle$. Show that this is a maximal compact subgroup in $Sp(2n,\mathbb{C})$, and find its dimension and rank (i.e. the dimension of a maximal torus T), its normalizer $N(T)$, and the corresponding Weyl group $W := N(T)/T$.

13. Write down miniversal deformations of each of the two quadratic hamiltonians $(p_1^2 + q_1^2) \pm (p_2^2 + q_2^2)$, and describe the subset in the parameter space consisting of *stable* hamiltonians, (i.e. consisting of *diagonalizable* infinitesimal symplectic transformations with purely imaginary spectrum).

14. Which of the following three groups are isomorphic, and which are not: (a) $Sp(2,\mathbb{R})$, (b) $SU(1,1)$ (the group of unimodular complex 2×2-matrices preserving the indefinite Hermitian form $|z_1|^2 - |z_2|^2$), (c) the group of *real* Möbius transformations $w = (az+b)/(cz+d)$ of the Riemann sphere?

15. Prove that the space of complex structures $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $J^2 = -E_{2n}$ in symplectic $\mathbb{R}^{2n}$ such that the symplectic form ω is the imaginary part of a (positive definite) Hermitian form (and hence such that $\omega(\cdot, J\cdot)$ is its real part) is contractible. (*Hint:* Identify the space with Siegel's "upper half-plane" $Sp(2n,\mathbb{R})/U_n$.)

16. Show that the structure of a real symplectic vector bundle can be upgraded to the structure of an Hermitian vector bundle (with the imaginary part of the Hermitian form being the given symplectic one). Derive that classification of real symplectic $2n$-dimensional vector bundles over a given base coincides with the classification of complex n-dimensional vector bundles over that base.

17. Prove "The Extension Theorem II" from page 26: A smooth field Ω of nondegenerate 2-forms on $T_N M$ whose restriction to TN define a closed differential 2-form ω on N can be extended to a symplectic structure to a neighborhood of N in M. (*Hint:* Use tubular neighborhood projection $\pi : M \to N$, and look for a differential 1-form α on M vanishing at the points of N and such that $d\alpha|_{T_N M} = \Omega - \pi^*\omega|_{T_N M}$, by applying a partition of unity on N to the RHS, and constructing α locally.)

18. Prove Moser's volume theorem: On a compact connected oriented manifold (without boundary) two volume forms with the same total volume can be transformed into each other by a diffeomorphism (and obviously *vice versa*.

19. The same — for two symplectic structures which can be connected by a continuous family of symplectic structures within the same De Rham cohomology class.

20. Use Moser's homotopy method to prove the Morse lemma: A germ of function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ at a non-degenerate critical point can be transformed by a germ of diffeomorphism $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ into the quadratic form of its 2nd differential $d_0^2 f$.

21. Establish a one-to-one correspondence between equivalence classes of complex vector bundles over a given manifold L and local symplectormorphism classes of isotropic embeddings of L into symplectic manifolds (more precisely, understanding the equivalence of two isotropic embeddings as a symplectomorphism between sufficiently small neighborhoods of L in the ambient symplectic manifolds which is identical on L).

22. Show that the *conormal* bundle N_X of a submanifold $X \subset M$ (by definition N_X is the subset in T^*M consisting of all covectors applied at a point of X and vanishing on tangent vectors to X at that point) is a Lagrangian submanifold in T^*M with respect to the canonical sumplectic structure. Compute the conormal bundle for the semicubical parabola $y^2 = x^3$ on the plane, first at non-nonsingular points, but then take the closure in $T^*\mathbb{R}^2$. Try to describe the resuting (singular) Lagrangian surface as an abstract algebraic variety (e.g. by characterizing somehow the ring of polynomial functions on it, or in any other way).

23. Compute adjoint and coadjoint orbits of the group of affine transformations on the line: $x \mapsto ax + b$.

24. Show that coadjoint orbits of the Lie algebra $\mathbb{R}^3$ with respect to the cross-product are concentric spheres $F = const$, where $F = (x^2 + y^2 + z^2)/2$ in Euclidean coordinates, and identify the symplectic (area) form on the orbits with the *Leray form* $(dx \wedge dy \wedge dz)/dF$ (an expression defining correctly a differential 2-form on the level sets of F at non-singular points).

25. When two vector fields u, v on a symplectic manifolds correspond to two closed 1-forms, the (exact) 1-form corresponding to their Lie bracket $[u, v]$ is $di_u i_v \omega$. Generalize this formula to the Lie bracket of two vector fields u and v corresponding to arbitrary (non-necessarily closed) differential 1-forms α and β.

26. Let v be a vector field on X. Its flow lifted naturally to T^*X preserves the canonical symplectic structure. Show that the Hamilton function of this flow is given by the rule: Its value at a point $p \in T_q^*X$ is equal to $p(v(q))$.

27. Prove that functions on T^*X, linear on the cotangent fibers, form a Lie subalgebra with respect to the Poisson bracket, which is canonically isomorphic to the Lie algebra of vector fields on the base.

28. Prove that a submanifold in a symplectic manifold is coisotropic if and only if the ideal of smooth functions vanishing on it is a Poisson subalgebra (i.e. closed with respect to the Poisson bracket).

29. Let $S \supset P$ be the germ of a symplectic leaf in a Poisson manifold P, and I the ideal of S in the algbera $C^\infty(P)$ of germs of smooth functions on P. Show that I (as well as every power I^k) is a Lie ideal with respect to the Poisson bracket, and that the induced Poisson bracket on I/I^2 endows the fibers of the (conormal to S) vector bundle $(T_SP/TS)^*$ with the structure of a Lie algebra (more precisely, the space of sections becomes a Lie algebra over the ring $C^\infty(S)$). Show that all Lie algebras in this family (over the base S) are isomorphic to each other (and to the linear approximation of the transverse Poisson structure of S).

30. Take ξ to be the Jordan normal form of subregular nilpotents in $\mathfrak{g}^* = gl^*_{n+1} \equiv gl_{n+1}$, and compute coadjoint orbits of the Lie algebra $\mathfrak{g}_\xi$ (the annihilator of ξ). Show (relying on the computations done in the lecture) that the foliation of $\mathfrak{g}^*_\xi$ into coadjoint orbits is not isomorphic to the transverse Poisson structure of ξ in $\mathfrak{g}^*$.

31. In the parameter plane (with coordinates c_2, c_3) of the family of surfaces $xy + z^3 + c_2 z + c_3 = 0$ (foliating the transversal Poisson manifold of the subregular nilpotent $\xi \in sl^*_3(\mathbb{C})$ into the closures of symplectic leaves), compute and sketch the discriminant, i.e. the image of the leaves of dimension 0.

32. Check that the De Rham diferential $d : \Omega^0(S^1) \to \Omega^1(S^1)$ is anti-symmetric with respect to the pairing $(f, \alpha) := \frac{1}{2\pi} \oint f\alpha$ (between functions f and 1-forms α), and hence defines a (translationally invariant) Poisson structure on Ω^1. Show that $c := (1, \alpha)$ is Casimir, and that $c = consts$ are the symplectic leaves. More specifically (assuming that the functions and 1-forms are complex-calued), show that the Poisson brackets between the Fourier coordinates $q_k := (\mathbf{e}^{-ikx}, \alpha)$, $p_k := (e^{ikx}, \alpha)$, $k = 1, 2, \ldots,$ are all zeros except $\{p_k, q_k\} = ik = -\{q_k, p_k\}$.

33. Prove that the fixed point locus of an anti-symplectic involution of a symplectic manifold is a Lagrangian submanifold (if non-empty).

34. Let a smooth map $\pi : (M^{2n}, \omega) \to X^n$ of a $2n$-dimensional compact symplectic manifold to an n-dimensional manifold be such that $pi^* C^\infty(X)$ is a commutative subalgebra in C^∞ with respect to the Poisson bracket, and let $X_0 \subset X$ be a (non-empty) connected component the regular value set of pi. Prove that $\pi^{-1} X^0 \xrightarrow{\pi} X_0$ is a Lagrangian fibration, and describe possible fibres of it.