Math H110. Fall'22. HW solutions

HW1

6. Let A, B, C be the centers of the circles, and A', B', C' the points moving along the circles. Then the radius-vector of the barycenter M' of the triangle $A'B'C'$ is

$$\overrightarrow{OM'} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) + \frac{1}{3}(\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}),$$

where the left average represents the radius-vector $\overrightarrow{OM}$ of the barycenter of the (time-independent) triangle ABC, and the right average is a fixed length R vector rotating with the the same angular velocity as the radii $\overrightarrow{AA'}, \overrightarrow{BB'}, \overrightarrow{CC'}$ of the circles. Thus, M' is moving along the circle of some radius R centered at M with that same angular velocity.

9. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be the vectors represented by the sides of a given triangle, $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$. Then $\mathbf{u} + \mathbf{v}/2, \mathbf{v} + \mathbf{w}/2, \mathbf{w} + \mathbf{u}/2$ are represented by the medians, which form a triangle since their sum is still $\mathbf{0}$. The medians of the latter triangle represent: $\mathbf{u} + \mathbf{v}/2 + (\mathbf{v} + \mathbf{w}/2)/2 = \mathbf{u} + \mathbf{v} + \mathbf{w}/4 = -3\mathbf{w}/4$, and respectively $-3\mathbf{u}/4$ and $-3\mathbf{v}/4$. Thus, they form a triangle with the sides parallel to the sides $\mathbf{w}, \mathbf{u}, \mathbf{v}$ of the initial triangle, but down-scaled by the factor $3/4$ (and oppositely directed).

22. Using $\overrightarrow{BC} = \overrightarrow{XC} - \overrightarrow{XB}$, etc., we find the given expression in the form

$$\overrightarrow{XA} \cdot \overrightarrow{XC} - \overrightarrow{XA} \cdot \overrightarrow{XB} + \overrightarrow{XB} \cdot \overrightarrow{XA} - \overrightarrow{XB} \cdot \overrightarrow{XC} + \overrightarrow{XC} \cdot \overrightarrow{XB} - \overrightarrow{XC} \cdot \overrightarrow{XA} = 0.$$

25. Let O denote the circumcenter of $\triangle ABC$. Then

$$XA^2 + XB^2 + XC^2 = (\overrightarrow{OX} - \overrightarrow{OA}) \cdot (\overrightarrow{OX} - \overrightarrow{OA}) +$$
$$(\overrightarrow{OX} - \overrightarrow{OB}) \cdot (\overrightarrow{OX} - \overrightarrow{OB}) + (\overrightarrow{OX} - \overrightarrow{OC}) \cdot (\overrightarrow{OX} - \overrightarrow{OC}) =$$
$$6R^2 - 2\,\overrightarrow{OX} \cdot (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) = 6R^2,$$

since in the case of a regular triangle, $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \mathbf{0}$.

27. Let us project the polyhedron to a plane ("screen") along the ("sight") direction of a unit vector $\mathbf{u}$. A face of the polyhedron with the pressure vector $\mathbf{f}$ projects to a polygon on the screen of signed area $\mathbf{f} \cdot \mathbf{u} = |\mathbf{f}| \cos\theta$ (because the angle between the plane of the face and the plane of the screen coincides with the angle θ between their normal vectors) and is negative whenever the line of "sight" enters the polyhedron through this face, and positive when it exits. Since each ray enters the polyhedron as many times as it exits it, the total signed area of the shadow of the polyhedron on the screen equals zero. Thus, for any unit vector $\mathbf{u}$, we have $(\sum \mathbf{f}_i) \cdot \mathbf{u} = 0$, implying $\sum \mathbf{f}_i = \mathbf{0}$.

HW2

60. The equation $23x^2 + 72xy + 2y^2 = 25$ can be brought to principal axes by rotating the coordinate system through the angle θ with $\mathbf{cot}\, 2\theta = (23-2)/72 = 7/24$, i.e. $\mathbf{cos}\, 2\theta = 7/25$, $\mathbf{sin}\, 2\theta = 24/25$ We have:

$$\begin{aligned} 23(X\cos\theta - Y\sin\theta)^2 + & 72(X\cos\theta - Y\sin\theta)(X\sin\theta - Y\cos\theta) + \\ 2(X\sin\theta + Y\cos\theta)^2 = & X^2(23\cos^2\theta + 72\sin\theta\cos\theta + 2\sin^2\theta) + \\ & Y^2(23\sin^2\theta - 72\cos\theta\sin\theta + 2\cos^2\theta) = \\ X^2(\frac{25}{2} + \frac{21}{2}\cos 2\theta + & 36\sin 2\theta) + Y^2(\frac{25}{2} - \frac{21}{2}\cos 2\theta - 36\sin 2\theta) = \\ & 50X^2 - 25Y^2 \end{aligned}$$

Thus, the curve is a hyperbola with the standard equation $2X^2 - Y^2 = 1$, or $X^2/\alpha^2 - Y^2/\beta^2 = 1$ with $\alpha = 1/\sqrt{2}$ and $\beta = 1$, in the rotatec coordinate system.

62. The curve $x^2 + xy + y^2 + \sqrt{2}(x - y) = 0$, after 45° rotation $x = (X-Y)/\sqrt{2}$, $y = (X+Y)/\sqrt{2}$ becomes $3X^2/2 + Y^2/2 - 2Y = 0$. Completing the square in Y, we get $3X^2/2 + (Y-2)^2/2 = 2$ or $3X^2/4 + \tilde{Y}^2/4 = 1$, which is an ellipse with semiaxes $2/\sqrt{3}$ and 2.

69. Write $ax^2 + 2bxy + cy^2 = y^2(at^2 + 2bt + c)$ where $t = x/y$. The quadratic function in t factors as $a(t - t_1)(t - t_2)$ (assuming $a \neq 0$) where $t_{1,2}$ are the roots, possibly non-real. Thus the quadratic form factors over $\mathbb{C}$ as the product $a(x - t_1 y)(x - t_2 y)$ of two lienear functions. It is $\pm$ the square of one linear function whenever the roots $t_{1.2}$ coincide (and hence real), i.e. the discrinmnant $4b^2 - 4ac = 0$. When $a = 0$, the discriminant vanishes exactly when $b = 0$, which is again the condition for $y(2bx + cy)$ to be a $\pm$ square of a linear form.

83. $\pm(X_1^2 + X_2^2 + X_3^2) = 0$ are "thick" points (the origin); $X_1^2 + X_2^2 - X_3^2 = 0$ and $X_1^2 - X_2^2 - X_3^2 = 0$ are cones (with the axis X_3 in the former case and X_1 in the latter); $X_1^2 + X_2^2 = 0$ and $-X_1^2 - X_2^2 = 0$ are "thick" lines (the X_3-axis); $X_1^2 - X_2^2 = 0$ is a pair of planes intersecting along the X_3-axis; $\pm X_1^2 = 0$ is the double plane, and $Q = 0$ is the whole space when Q is identically zero.

98. $4ac - 4b^2 = (a + c)^2 - (a - c)^2 - (2b)^2 = 0$ becomes the standard cone $X^2 + Y^2 = Z^2$ after the linear change of coordinates $X = (a - c)$, $Y = 2b$, $Z = (a + c)$.

HW3

147. Any 4 elements in a 3-diemensional space (of polynomials $ax^2 + bx + c$) are linearly dependent. Here is one way to check independence of any three of them: Assume that a linear combination (say of the last three) vanishes, $\alpha(x-1)^2+\beta(x-2)^2+\gamma(x-3)^2 = 0$, then plug $x = 1, 2, 3$ to conclude that $\beta+4\gamma = 0$, $\alpha+\gamma = 0$, $4\alpha+\beta = 0$, and derive from these that $\alpha = \beta = \gamma = 0$. Alternatively, one that the basis $1, x, x^2$ of the space is expressible via linear combinations of (in this example) $(x-1)^2, (x-2)^2, (x-3)^2$. Namely, $(x-1)^2 - (x-2)^2 = 2x-3$, $(x-2)^2 - (x-3)^2 = 2x-5$, and the difference if 2, which already shows that 1 and x are expressible; after that $(x-1)^2 = x^2 - 2x + 1$ shows that x^2 is expressible too.

150. The subspace can be interpreted as the graph of linear map $(x_2, x_3) \mapsto (x_1, x_4)$ given by $x_1 = -x_2-x_3$, $x_4 = -x_2-x_3$, and has therefore dimension 2 (equal to that of the domain of the map), with a basis obtained by lifting the basis $(x_2, x_3) = (1, 0)$ and $= (0, 1)$ in the domain to the graph: $(x_1, x_2, x_3, x_4) = (-1, 1, 0, -1)$ and $= (-1, 0, 1, -1)$ respectively.

154. $E_{\mathbf{v}} \in \mathcal{V}^{**}$ because $E_{\mathbf{v}}(\lambda\mathbf{f}+\mu\mathbf{g}) = \lambda\mathbf{f}(\mathbf{v})+\mu\mathbf{g}(\mathbf{v}) = \lambda E_{\mathbf{v}}(\mathbf{f})+\mu E_{\mathbf{v}}(\mathbf{g})$ (i.e. $E_{\mathbf{v}}$ is a linear form on $\mathcal{V}^*$ due to the pointwise nature of the operations with functions $\mathbf{f}, \mathbf{g} \in \mathcal{V}^*$). $E : \mathcal{V} \mapsto \mathcal{V}^{**}$ is linear because $E_{\lambda\mathbf{u}+\mu\mathbf{v}}(\mathbf{f}) = f(\lambda\mathbf{u}+\mu\mathbf{v}) = \lambda\mathbf{f}(\mathbf{u})+\mu\mathbf{f}(\mathbf{v}) = \lambda E_{\mathbf{u}}(\mathbf{f})+\mu E_{\mathbf{v}}(\mathbf{f})$, (i.e. $E_{\lambda\mathbf{u}+\mu\mathbf{v}} = \lambda E_{\mathbf{u}} + \mu E_{\mathbf{v}}$ due to the lineariry of $\mathbf{f} : \mathcal{V} \to \mathbb{K}$).

157. The subspace is spanned by two (non-proportional, and hence linearly independent) functions $\mathbf{cos}\, x$ and $\mathbf{sin}\, x$, because

$$\mathbf{cos}(x+\theta) = \mathbf{cos}\,\theta\,\mathbf{cos}\,x - \mathbf{sin}\,\theta\,\mathbf{sin}\,x$$

is their linear combination. The subspace has dimension 2 indeed provided that there is at least one pair among θ_i such that $\theta_i - \theta_j$ is not an integer multiple of π. (Otherwise all the n functions are proportional to each other.)

159. The composition $\mathbf{f} \circ \pi$ of a linear form $\mathbf{f} : \mathcal{V}/\mathcal{W} \to \mathbb{K}$ with the canonical projection $\pi : \mathcal{V} \to \mathcal{V}/\mathcal{W}$ is a linear form on $\mathcal{V}$ vanishing on $Ker\pi = \mathcal{W}$ (i.e. lying in $\mathcal{W}^{\perp}$). Colversely, $\mathbf{g} \in \mathcal{W}^{\perp}$ is a linear form $\mathcal{V} \to \mathbb{K}$ vanishing on $\operatorname{Ker}\pi$, hence constant on each affine subspace $\mathbf{v} + \mathcal{W}$, and therefore descending to a linear form $\mathbf{f} : \mathcal{V}/\mathcal{W} \to \mathbb{K}$ such that $\mathbf{g} = \mathbf{f} \circ \pi$.

HW4

173. A matrix $[a_{ij}]$ of a linear map A is upper-triangular iff (="if and only if") in each column j, the entries $a_{ij} = 0$ when $i > j$. In other words, for each j, $A\mathbf{e}_j$ is a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_j$, i.e. lies in the subspace $Span(\mathbf{e}_1, \dots, \mathbf{e}_j)$. This is equivalent to saying that these subspaces are A-invariant. If two linear maps A and B map a subspace to itself, then so does their composition AB. Thus, the product of upper-triangular matrices is upper-triangular.

176. The given matrix represents the counter-clockwise rotation on the plane through the angle of 19°. The 19th power of the matrix represents the 19th consecutive iteration of this rotation, i.e. the rotation through $19 \cdot 19° = 361°$ which modulo the full turn is the counter-clockwise rotation through the angle of 1°: $\begin{bmatrix} \cos 1° & -\sin 1° \\ \sin 1° & \cos 1° \end{bmatrix}$.

178. The kth iteration of N maps $\mathbf{e}_1 \mapsto \mathbf{e}_{k+1}$, $\mathbf{e}_2 \mapsto e_{k+2}$, etc., up to $\mathbf{e}_{n-k} \mapsto \mathbf{e}_n$, while all $\mathbf{e}_j$ with $j > n - k$ are already mapped to $\mathbf{0}$. Thus, $N^0 = I$ has 1s on the principal diagonal and all other entries equal to o, while the matrix of N^k, $k = 1, 2, \dots$ has 1s on the "kth diagonal above the principal one", and all other entries equal to 0. In particular, starting from $k = n$, $N^k = 0$.

189. For $B = 2x_1(y_1 + y_2)$, $S := (B + B^t)/2 = x_1(y_1 + y_2) + y_1(x_1 + x_2) = 2x_1y_1 + x_1y_2 + x_2y_1$, and $A := (B - B^t)/2 = x_1(y_1 + y_2) - y_1(x_1 + x_2) = x_1y_2 - x_2y_1$.

195. $(x_1 + \dots + x_n)(y_1 + \dots + y_n)$ is a symmetric bilinear form whose "restriction to the diagonal" $\mathbf{y} = \mathbf{x}$ is $(x_1 + \dots + x_n)^2$ as required. $\sum_{i<j} x_ix_j = \sum_{i \neq j} x_ix_j/2$ (i.e. has the symmetric coefficient matrix all of whose diagonal entries are zeros and off diagonal entries equal to 1/2). The corresponding symmetric bilinear form is therefore $\sum_{i \neq j} x_iy_j/2$.

HW5

197. For $T = \bar{z}_1 w_2$ we have $T^\dagger = \bar{z}_2 w_1$. Consequently

$$\bar{z}_1 w_2 = \frac{\bar{z}_1 w_2 + \bar{z}_2 w_1}{2} + i\frac{\bar{z}_1 w_2 - \bar{z}_2 w_1}{2i}.$$

The corresponding Hermitian quadratic forms are $(\bar{z}_1 z_2 + \bar{z}_2 z_1)/2$ and $(\bar{z}_1 z_2 - \bar{z}_2 z_1)/2i$ i.e. the real and imaginary parts of $\bar{z}_1 z_2$.

222. Note that the equation is a polynomial in x of degree not exceeding n. When x is equal to one of a_i, the matrix has two identical rows, and hence zero determinant. Thus, $x = a_1, \dots, a_n$ are distinct roots of the polynomial. It remains to show that the polynomial is not identically zero (in which case no more roots are possible), i.e. that the determinant does not vanish whenever x differs from all a_i. This can be done by induction on n. Namely, the top coefficient of the polynomial is, up to a sign, the same determinant of the previous size, and is non-zero by the induction hypothesis as long as all a_i are distinct. The base of induction is also easy to establish.

223. $\det A = \det A^t = \det(-A) = (-1)^n \det A = -\det A$ when n is odd.

225. Under a change of variables $\mathbf{x} = C\mathbf{x}'$, the symmetric matrix Q of a quadratic form is transformed into $Q' = C^t QC$. Therefore $\det Q' = \det Q (\det C)^2$ and has the same sign as $\det Q$ (since $\det C \neq 0$).

233. Since $(\operatorname{adj}(A))A = (\det A)I_n$, we have $(\det A)\det(\operatorname{adj}(A)) = (\det A)^n$. Therefore $\det(\operatorname{adj}(A)) = (\det A)^{n-1}$ provided that $\det A \neq 0$. When $\det A = 0$ this formula also holds, because the adjugate matrix is not invertible (for otherwise $A = 0$ and $\operatorname{adj}(A) = 0$ — contradiction).

HW6

273. We have $y_1 + y_2 + y_3 = 0$, i.e. the rank of A is at most 2. Taking $\mathbf{e}_1 = (1,0,0)^t$ and $\mathbf{e}_2 = (0,1,0)^t$, we find their images to be $\mathbf{f}_1 = (2,-1,-1)^t$ and $\mathbf{f}_2 = (-1,2,1)^t$, which are clearly non-proportional to each other, and hence form a basis in the range of A (which therefore has dimension 2 indeed). The kernel of A is spanned by $\mathbf{e}_3 = (1,1,1)$. Picking on the role of $\mathbf{f}_3$ any vector not lying in the plane $y_1 + y_2 + y_3 = 0$ (e.g. $\mathbf{f}_3 = (0,0,1)^t$), we now have two bases: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the source space, and $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ the target space, in which A has the matrix E_2 (i.e. $A\mathbf{e}_1 = \mathbf{f}_1$, $A\mathbf{e}_2 = \mathbf{f}_2$, $A\mathbf{e}_3 = \mathbf{0}$. The system $A\mathbf{x} = \mathbf{b}$ is consistent whenever $\mathbf{b}$ lies in the range, i.e. satisfies $b_1 + b_2 + b_3 = 0$.

276. Clearly the given 3×4-matrix has non-zero 2×2-minors (e.g. the one in the left upper corner is equal to -11). In fact all the four 3×3-minors vanish (which can be checked by a tedious yet straightforward computation), implying that the rank of the matrix equals 2.

279. (a) Yes: the solution set $A^{-1}\mathbf{b}$ is an affine subspace parallel to $\operatorname{Ker} A$, and consists of one point only if $\operatorname{Ker} A = \{0\}$ (indicating that A is injective). (b) Yes: since A is injective, the dimension of its range is the same as the dimension of the source space. (c) Yes: the dimension m of the target space cannot be less than that of the range. (d) No: when the dimension m of the target space > 2021, A is not surjective and hence not invertible. (e) Yes: when $A^tA\mathbf{x} = \mathbf{0}$, we have the dot-product $\langle A\mathbf{x}, A\mathbf{x}\rangle = \langle A^tA\mathbf{x}, \mathbf{x}\rangle = 0$, and hence $A\mathbf{x} = \mathbf{0}$. (This is the only point where we use that the field $\mathbb{K} = \mathbb{R}$, in which case only the zero vector has zero inner square.) Since A is injective, this implies $\mathbf{x} = 0$, i.e. the 2021×2021-matrix A^tA has zero kernel, and is therefore invertible. (f) No: when $m > 2021$, the $m \times m$-matrix of the composition $\mathbb{R}^m \stackrel{A^t}{\to} \mathbb{R}^{2021} \stackrel{A}{\to} \mathbb{R}^m$ is guaranteed to have rank $\leq 2021 < m$, and therefore has zero determinant. (g) No: when $m > 2021$, the m rows, i.e. m linear forms on $\mathbb{R}^{2021}$ must be linearly dependent in $\mathbb{R}^{2021*}$. (h) Yes: the rank of A equals 2021, i.e. its 2021 columns must be linearly independent.

281. Let the planes be $\mathbf{u}_0 + \mathcal{U}$ and $\mathbf{v}_0 + \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are two 2-dimensional linear subspaces. Then $\mathbf{v}_0 + \mathcal{U}$ and $\mathbf{v}_0 + \mathcal{V}$ have a common point $\mathbf{v}_0$ and lie in the affine subspace $\mathbf{v}_0 + \mathcal{U} + \mathcal{V}$ parallel to $\mathcal{U} + \mathcal{V}$ (whose dimension ≤ 4). Then $\mathbf{u}_0 + \mathcal{U}$ and $\mathbf{v}_0 + \mathcal{V}$ lie in the affine subspace $\mathbf{v}_0 + \mathcal{W}$, where $\mathcal{W}$ is spanned by $\mathcal{U} + \mathcal{V}$ and $\mathbf{u}_0 - \mathbf{v}_0$ (and has dimension ≤ 5). Indeed, since $\mathcal{W}$ contains $\mathcal{V}$, clearly $\mathbf{v}_0 + \mathcal{W}$ contains $\mathbf{v}_0 + \mathcal{V}$. But $\mathcal{W}$ also contains $\mathbf{u}_0 - \mathbf{v}_0 + \mathcal{U}$, and hence $\mathbf{v}_0 + \mathcal{W}$ contains $\mathbf{u}_0 + \mathcal{U}$.

286. Given subspaces $\mathcal{V}, \mathcal{W} \subset \mathbb{K}^n$ of dimensions k and l and such that $\dim \mathcal{V} | cap \mathcal{W} = d$, the subspace $\mathcal{V} + \mathcal{W}$ has dimension $k + l - d$ (by the dimension counting formula). Consequiently, picking a basis $\mathbf{e}_1, \dot{;} \mathbf{e}_d$ in $\mathcal{V} \cap \mathcal{W}$, comleting it by $\mathbf{e}_{d+1}, \ldots, \mathbf{e}_k$ to a basis in $\mathcal{V}$ and by $\mathbf{e}_{k+1}, \ldots, \mathbf{e}_{k+l-d}$ to a basis in $\mathcal{W}$, we obtain a basis in $\mathcal{V} + \mathcal{W}$, which then can be completed by $\mathbf{e}_{k+l-d+1}, \ldots, \mathbf{e}_n$ to a basis in $\mathbb{K}^n$. Therefore all pairs $(\mathcal{V}, \mathcal{W})$ of subspaces in KK^n of given dimensions $k, l \leq n$ and given dimension d of their intersection, $\max(0, k + l - n) \leq d \leq \min(k, l)$, are equivalen to each other (and obviously *vice versa*: when to pairs of subspaces are equivalent, their respective dimensions and the dimensions of their intersections coincide).

HW7

288. The answers are found in the *Hints, answers, index* section. Let's hope they are correct.

289.

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 7 & -4 & 11 & \lambda \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -.5 & .5 & .5 & .5 \\ 0 & 2.5 & -1.5 & 3.5 & 1.5 \\ 0 & 7.5 & -4.5 & 10.5 & \lambda - .5 \end{array}\right]$$
$$\mapsto \left[\begin{array}{cccc|c} 1 & -.5 & .5 & .5 & .5 \\ 0 & 1 & -.6 & 1.4 & .6 \\ 0 & 0 & 0 & 0 & \lambda - 5 \end{array}\right].$$

Thus, the system is consistent only when $\lambda = 5$.

290 (e).

$$A = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & .5 & 1.5 & -1.5 \\ 0 & -2.5 & -2.5 & 1.5 \\ 0 & 2.5 & 2.5 & -1.5 \\ 0 & -5 & -5 & 3 \end{bmatrix} \mapsto$$
$$\begin{bmatrix} 1 & .5 & 1.5 & -.5 \\ 0 & 1 & 1 & -.6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & -.2 \\ 0 & 1 & 1 & -.6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A'.$$

Therefore, the matrix A has rank 2, the two leftmost columns of A form a basis in its column space, and the two non-zero rows of the reduced row-echelon form A' form a basis in the row space of A. Note that the homogeneous system $A\mathbf{x} = \mathbf{0}$ is equivalent to $A'\mathbf{x} = \mathbf{0}$ which has the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t_1 + .2t_2 \\ -t_1 + .6t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} .2 \\ .6 \\ 0 \\ 1 \end{bmatrix}.$$

The two columns featured in the answer form a basis in the kernel of A.

291 (a). The answer in the book is incorrect. In fact $\det A = -7$, and

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & 8 & 3 \\ 1 & 1 & 3 \\ -1 & 6 & 4 \end{bmatrix}.$$

237. We apply Laplace's theorem to the 4×4-matrix formed by two copies of the given 2×4-matrix: one in the top and one in the bottom two rows. On the one hand, the determinant equals 0 since the matrix has identical rows. On the other hand, it is equal to $P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} + P_{23}P_{14} - P_{24}P_{13} + P_{34}P_{12}$. Thus, $0 = 2(P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23})$ as required.

HW8

306. Q is obtained by composing the quadratic form

$$\mathbb{R}^{p+q} \to \mathbb{R} : Q'(\mathbf{y}) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2$$

with the linear map

$$\mathbb{R}^n \stackrel{A}{\to} \mathbb{R}^{p+q} : y_1 = \mathbf{a}_1(\mathbf{x}), \ldots, y_p = \mathbf{a}_p(\mathbf{x}), y_{p+1} = \mathbf{b}_1(\mathbf{x}), \ldots, y_{p+q} = \mathbf{b}_q(\mathbf{x}).$$

A subspace $\mathcal{V}^k \subset \mathbb{R}^n$ on which Q is positive definite must map injectively to $\mathbb{R}^{p+q}$ and have Q' positive definite on $A(\mathcal{V})$. Thus $k = \dim \mathcal{V} = \dim A(\mathcal{V}) \leq p$, i.e. the positive inertia index of Q does not exceed p. The argument for the negative inertia index being $\leq q$ is similar.

307. $(x_1 + x_3)x_2 = \pm 1$ are equivalent to $y_1 y_2 = \pm 1$ in $\mathbb{R}^3$, i.e. are both the cylinders over hyperbolas. $x_1x_2 + x_2x_3 + x_3x_1 = x_2(x_1 + x_3) + x_3x_1 \pm 1$ is equivalent to $y_2y_3 + y_3y_1 - y_1^2 = \pm 1$ (where $y_1 = x_1, y_2 = x_2, y_3 = x_1 + x_3$). By completing squares, we find:

$$\pm 1 = -(y_1 - y_3/2)^2 + \frac{y_3^2}{4} - y_2y_3 = -(y_1 - y_3/2)^2 + \frac{(y_3 - 2y_2)^2}{4} - y_3^2.$$

Thus, the conics are equivalent to $z_1^2 - z_2^2 - z_3^2 = \pm 1$ (where $z_1 = y_2/2$, $z_2 = y_1 - y_3/2$, and $z_3 = (y_3 - 2y_2)/2$), i.e. are the hyperboloids of two $(+)$ and one $(-)$ sheets.

308. The "hyperboloids" in $\mathbb{R}^4$ have normal forms $x^2 + y^2 + z^2 + u^2 = 1$ (the sphere, one sheet), $x^2 + y^2 + z^2 + u^2 = -1$ (the empty set, no sheets), $x^2 + y^2 + z^2 = 1 + u^2$ (which is a family of spheres in the affine subspaces $u = const$ of radiuses $\sqrt{1 + u^2}$ (non-empty for each u and hence forming a connected hypersurface — one sheet), $x^2 + y^2 + z^2 = u^2 - 1$ (two families of spheres of radiuses $\sqrt{u^2 - 1}$, one for $u \geq 1$, one for $u \leq -1$ — two connected components, and $x^2 + y^2 - z^2 = 1 + u^2$ — which is actually connected (one sheet), because it is a family of one-sheeted hyperboloids in the hyperplanes $u = const$, non-empty for each value of u.

311. From the classification theorem, we have: $z_1^2 + z_2^2 = 1$ (complex "circle" = "hyperbola"); $z_1^2 = z_2$ (parabola), or the cylinder over a conic in $\mathbb{C}^1$, i.e. $z_1^2 = 1$ (two parallel lines in $\mathbb{C}^2$ or $z_1^2 = 0$ (two merged lines in $\mathbb{C}^2$).

314. The classification theorem of conics in $\mathbb{C}^n$ implies that the numbers N_n of the equivalence classes satisfy $N_n = 3 + N_{n-1}$, where $N_1 = 2 = 3 - 1$ (from Exercise 311). By induction, $N_n = 3n - 1$.

HW9

325. For the symmetric bilinear form $Q(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t Q \mathbf{y}$, construct inductively the Q-orthogonal basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$ with $\mathbf{f}_k = \mathbf{e}_k + \sum_{i<k} c_{ik}\mathbf{e}_i$, where the coefficients c_{ik}, $i < k$, are found so that $\mathbf{f}_k$ is Q-orthogonal to $\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}$ (i.e. as in the proof of Sylvester's Rule, but not normalizing the values $Q(\mathbf{f}, \mathbf{f}_i)$ to ± 1 by rescaling the vectors $\mathbf{f}_k$). Together with $c_{kk} = 1$ and

$c_{ik} = 0$ for $i > k$, they form a *unipotent* upper-triangular matrix C such that $C^tQC = D$, where D is diagonal with $Q(\mathbf{f}_i, \mathbf{f}_i)$ on the diagonal. The leading minors do not change under transformations $Q \mapsto C^tQC$ defined by unipotent upper-triangular matrices C. Consequently, $Q(f_i, \mathbf{f}_i) = \Delta_i/\Delta_{i-1}$ where Δ_i, $i = 0, 1, \ldots, n$, are the leading minors of Q.

328. After multiplication by 2, the matrix of the quadratic form has 2s on the principal diagonal, -1s right above and right below it, and 0s everywhere else. The determinant Δ_n of this matrix can be computed inductively from $\Delta_1 = 2$, $\Delta_2 = 3$, and $\Delta_n = 2\Delta_{n-1} - \Delta_{n-2}$ (cofactor expansion), implying that $\Delta_n = n+1$. Consequently, the leading minors of the quadratic form are all positive (equal to $(1), 2, 3, \ldots, n+1$), showing that the form is positive definite.

329. Note that the determinant of the coefficient matrix of a quadratic form, under a change of coordinates, is multiplied by the square of the determinant of the transition matrix. However, when the quadratic form changes, the determinant increases or decreases regardless of the coordinate system, since the transition matrix stays the same. So, preparing to apply the unipotent triangular change of coordinates from exercise 325 — which expresses the determinant of the coefficient matrix as the product of $Q(\mathbf{f}_i)$ — we begin with a coordinate system where the linear form l whose square is added to Q is taken on the role of the last coordinate. Then all the basis vectors $\mathbf{e}_i$ except the last one lie in the hyperplane $l(\mathbf{x}) = 0$ where $Q(\mathbf{x}) + l(\mathbf{x})^2$ coincides with $Q(\mathbf{x})$. Consequently all but the last one of the vectors $\mathbf{f}_i$ of the Q-orthogonal basis (obtained as in exercise 325) are the same for Q and for $Q + l^2$ (and satisfy $l(\mathbf{f}_i) = 0$). Therefore for $i < n$ the (positive) diagonal factors are the same: $Q(\mathbf{f}_i) = Q(\mathbf{f}_i) + l(\mathbf{f}_i)^2$, while $Q(\mathbf{f}_n) < Q(\mathbf{f}_n) + l(\mathbf{f}_n)^2$. Thus, their product increases when Q is replaced with $Q + l^2$.

356. For an $m \times n$ matrix $A = [a_{ij}]$, the trace of the $n \times n$ matrix $A^\dagger A$ is equal to $\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2$, which is positive unless all $a_{ij} = 0$. Thus, if $\sum A_i^\dagger A = 0$, then $\sum \operatorname{tr} A_i^\dagger A = 0$, and therefore all $A_i = 0$.

364. $\langle A\mathbf{x}, A\mathbf{x}\rangle = \langle A^\dagger\mathbf{x}, A^\dagger\mathbf{x}\rangle$ for all $\mathbf{x} \in \mathcal{V}$ is equivalent to $\langle \mathbf{x}, A^\dagger A\mathbf{x}\rangle = \langle \mathbf{x}, AA^\dagger\mathbf{x}\rangle$ for all $\mathbf{x} \in \mathcal{V}$, and is equivalent to $\langle \mathbf{x}, S\mathbf{x}\rangle = 0$ for all $\mathbf{x} \in \mathcal{V}$ where $S = i(A^\dagger A - AA^\dagger)$. This is equivalent to $S = 0$, i.e. to $A^\dagger A = AA^\dagger$. Indeed, a function of the form $\mathbf{x} \mapsto \langle \mathbf{x}, S\mathbf{x}\rangle$ is an Hermitian quadratic form (Hermitian-symmetric in our case, though this is curremtly irrelevant) which uniquely determines the sesquilinear form $(\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, S\mathbf{y}\rangle$, which is therefore zero in our case, implying $S = 0$. Conversely, normality of A implies $S = 0$ and hence $\langle A\mathbf{x}, A\mathbf{x}\rangle = \langle A^\dagger\mathbf{x}, A^\dagger\mathbf{x}\rangle$ for all $\mathbf{x}$.

HW10

370. Since A is normal, we have (by the Spectral Theorem) the decomposition $\mathcal{V} = \oplus_\lambda \mathcal{W}_\lambda$ of the space $\mathcal{V}$ into a direct orthogonal sum of the eigenspaces of A. Each $\mathcal{W}_\lambda$ is also the eigenspace of $A^\dagger$ with the eigenvalue $\bar{\lambda}$. If $AB = BA$ then $B^\dagger A^\dagger A^\dagger B^\dagger$. Therefore each $\mathcal{W}_{|lambda}$ is both $B-$ and $B^\dagger$-invariant. Therefore B, being normal in $\mathcal{V}$ (i.e. commuting with $B^\dagger$), also acts on $\mathcal{W}_\lambda$ as a normal operator. Thus, by the Spectral Theorem applied to $B|_{\mathcal{W}_\lambda}$, each $\mathcal{W}_\lambda$ decomposes into the direct orthogonal sum of eigenspaces of $B|_{\mathcal{W}_\lambda}$. [If more commuting normal operators are given, this process can be continued inductively to decompose $\mathcal{V}$ into a direct orthogonal sum of common eigenspaces of the operators.] Combining orthonormal bases — one in each of the common eigenspaces of the operators — yields an orthonormal basis in $\mathcal{V}$ consisting of the operators' eigenvectors.

374. If $P^2 = P$, then every $\mathbf{x} \in \mathcal{V}$ can be uniquely written as the sum $\mathbf{x} = P\mathbf{x} + (I - P)\mathbf{x}$, where $P\mathbf{x}$ lies in the eigenspace $\mathcal{V}_1$ of P with the eigenvalue 1, and $(I - P)\mathbf{x}$ in the eigenspace $\mathcal{V}_0$ f P with the eigenvalue 0. The operator is therefore the projector to $\mathcal{V}_1$ along $\mathcal{V}_0$. But these eigenspaces are orthogonal to each other if and only if $P^\dagger = P$.

383. According to Exercise 328 (from the previous homework), the quadratic form $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive definite. Inner products of basis vectors $\mathbf{e}_i$ are: $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 2$ (i.e. $|\mathbf{e}_i| = \sqrt{2}$), $\langle \mathbf{e}_i, \mathbf{e}_{i+1} \rangle = -1$ (i.e. the cosine of the angle between these vectors is $-1/\sqrt{2}^2 = -1/2$ and hence the angle is 120°), and $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ when $|i - j| > 1$ (i.e. the angle between such vectors is 90°).

399. The answer is found on page 174 of the text.

401(b). The coefficient matrix $S = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$ has the characteristic polynomial $\lambda^3 - 3\lambda^2 - 6\lambda + 8$ with an obvious root $lambda = 1$, and factors as

$$(\lambda - 1)(\lambda^2 - 2\lambda - 8) = (\lambda - 1)(\lambda - 4)(\lambda + 2).$$

The eigenvectors corresponding to the eigenvalues $\lambda_{1,2,3} = 4, 1, -2$ are found by solving the systems $S\mathbf{x}_i = \lambda_i \mathbf{x}_i$: $\mathbf{x}_1 = (2, -2, 1)^t$, $\mathbf{x}_2 = (2, 1, -2)^t$, and $\mathbf{x}_3 = (1, 2, 2)^t$, which all have length 3. Therefore after the orthogonal change of coordinates $\mathbf{x} = U\mathbf{y}$, where $U = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ the coefficient matrix will become diagonal: $U^t S U = \text{diag}(4, 1, -2)$, and the quadratic form $4y_1^2 + y_2^2 - 2y_3^2$ as required.

HW11

403. By the real version of the Spectral Theorem, any invertible anti-symmetric transformation Ω can be described as the superposition of non-zero anti-symmetric transformations $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$ in the direct orthogonal sum of Euclidean planes RR^2. Therefore, in a suitable (Cartesian) coordinate system the non-degenerate anti-symmetric bilinear form is written as

$$\mathbf{x}^t\Omega\mathbf{y} = \omega_1(x_1y_2 - x_2y_1) + \cdots + \omega_n(x_{2n-1}y_{2n} - x_{2n}y_{2n-1}),$$

The coefficients $\omega_1, \ldots, \omega_n$ can be assumed positive (because transposing the coordinates on the ith plane changes the sign of ω_i). Rescaling the coordinates on each plane by $\sqrt{\omega_i}$ one obtains a unique normal form $\sum_i(x_{2i-1}y_{2i} - x_{2i}y_{2i-1})$ of a non-degenerate anti-symmetric bilinear form in $\mathbb{R}^{2n}$. In the odd dimension, anti-symmetric bilinear forms are necessarily degenerate, which follows from the Spectral Theorem, but is also known from the fact that an anti-symmetric matrix of an odd size has zero determinant.

405. Let P_u and P_v be the orthogonal projectors from $\mathbb{R}^4$ to the planes $\mathcal{U}$ and $\mathcal{V}$ respectively. [BTW, an orthogonal projector P satisfy P satisfies $P^2 = P$, has two perependicular eigenspaces with the eigenvalues 1 and 0, and is cosequently self-adjoint ($P^t = P$) and positive: $\langle \mathbf{x}, P\mathbf{x}\rangle = \langle \mathbf{x}, P^2\mathbf{x}\rangle = \langle P\mathbf{x}, P\mathbf{x}\rangle \geq 0$ for all $\mathbf{x}$.] For $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, since $P_v = I$ on $\mathcal{V}$, we have $\langle \mathbf{x}, T\mathbf{y}\rangle = \langle \mathbf{x}, P_vP_u\mathbf{y}\rangle = \langle P_v\mathbf{x}, P_u\mathbf{y}\rangle = \langle \mathbf{x}, P_u^2\mathbf{y}\rangle = \langle P_u\mathbf{x}, P_u\mathbf{y}\rangle$. Consequently $\langle \mathbf{x}, T\mathbf{y}\rangle = \langle \mathbf{y}, T\mathbf{x}\rangle$, i.e. $T^t = T$, and $\langle \mathbf{x}, T\mathbf{x}\rangle \geq 0$, i.e. T is positive. By the Spectral Theorem for real self-adjoint operators, T has an orthonormal basis of eigenvectors $\mathbf{e}_1, e_2 \in \mathcal{V}$ with eigenvalues $\lambda_1, \lambda_2 \geq 0$: $P_vP_u\mathbf{e}_i = \lambda_i\mathbf{e}_i$. Since each orthogonal projection can only decrease the length of a vector, we conclude that $0 \leq \lambda_i \leq 1$ and thus each λ_i is the cosine of some angle between 0 and $\pi/2$. Note that $\langle P_u\mathbf{e}_2, P_u\mathbf{e}_1\rangle = \langle \mathbf{e}_2, T\mathbf{e}_1\rangle = \lambda_1\langle \mathbf{e}_2, \mathbf{e}_1\rangle = 0$, i.e. the images $P_u\mathbf{e}_i$ of the vectors $\mathbf{e}_i$ in $\mathcal{U}$ are orthogonal, and $\lambda_i = \mathbf{cos}^2\,\theta_i$, where θ_i is the angle in $\mathbb{R}^4$ between $\mathbf{e}_i \in \mathcal{V}$ and $P_u\mathbf{e}_i \in \mathcal{U}$.

408. An ellipsoid $E \subset \mathbb{R}^3$ in a suitable Cartesian coordinate system is given by the equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where we may assume that the semiaxes satisfy $a \leq b \leq c$. If a section of E by a plane passing through the origin is a circle of radius r, then Cauchy's interlacing theorem shows that $r \leq b \leq r$, i.e. $r = b$ (the middle semiaxis of E). A secting plane P containing the y-axis is symmetric (together with E) under the reflection $(x, y, z) \mapsto (x, -y, z)$. Therefore the ellipse $P \cap E$ is symmetric about the line $P \cap \{y = 0\}$, and hence about the line $x = z = 0$ (the middle axis of E) perpendicular to it. Thus these two lines are the principal axes of the ellipse $P \cap E$, which is therefore a circle if and only if its semiaxis lying in the plane $y = 0$ has length b. To find such P, note that the intersection of E with the plane $y = 0$ is the ellipse $x^2/a^2 + z^2/c^2 = 1$, and when $a < b < c$, it contains 2 pairs of centrally symmetric points of distance b from the origin.

The two planes spanned by any of these points and the y-axis intersect the ellipse in a circle of radius b. In the extreme cases $b = c$ or $b = a$ the section of E by the plane $x = 0$ or respectively $z = 0$ is the required circle as well.

410. It follows by induction on codimension k of the subspace from the Cauchy interlacing theorem. If (by the induction hypothesis) $\alpha_i \le \alpha'_i \le \alpha_{i+k}$ for all $i = 1, \dots, n-k$, then for a hyperplane in the subspace of codimension k, the smiaxes α''_i, $i = 1, \dots, n-k-1$, of the corresponding ellipsoid we have: $\alpha_i \le \alpha'_i \le \alpha''_i \le \alpha'_{i+1} \le \alpha_{i+1+k}$.

415. Let T denote the cyclic shift operator studied on page 174. The matrix defined by the periodic sequence $\{C_k\}$ is $C := C_1 T + C_2 T^2 + \cdots + C_n T^n$. Since $T^t = T^{-1}$ and all polynomials in T and T^{-1} commute with each other, we conclude that C is normal, that all such operators (corresponding to different periodic sequeneces) commute, and the Fourier basis in $\mathbb{C}^n$ described on page 174 is the common orthonormal basis of eigenvectors of the operators C. The respective eigenvalues are $C_1\zeta + C_2\zeta^2 + \cdots + C_n\zeta^n$ where $\zeta = e^{2\pi\sqrt{-1}l/n}$, $l = 1, \dots, n$.

HW12

420. The operator $\mathbf{x} \mapsto \mathbf{a}(\mathbf{x})\mathbf{v}$ has rank 1. Its range is spanned by $\mathbf{v} \neq \mathbf{0}$, and the kernel is the hyperplane given by the equation $\mathbf{a}(\mathbf{x}) = \mathbf{0}$. When $\mathbf{a}(v) \neq 0$, the space is the direct sum of the kernel, which is the eigenspace with the eigenvalue 0, and the range, which is the eigenline with the eigenvalue $\mathbf{a}(v)$. When $\mathbf{a}(\mathbf{v}) = 0$, the operator is nilpotent (its square equals 0) with its kernel being the only eigenspace. (When $\dim \mathcal{V} = 1$, the 2nd possibility does not occur.)

423. Any linear combination $A = C_0N^0 + C_1N^1 + \cdots + C_{n-1}N^{n-1}$ of the non-zero powers of a regular nilpotent $N : \mathbb{K}^n \to \mathbb{K}^n$ commutes with N. It is not hard to show that any operator $A : \mathbb{K}^n \to \mathbb{K}^n$ commuting with N is of this form. Namely, if N acts by $\mathbf{e}_n \mapsto \mathbf{e}_{n-1} \mapsto \cdots \mapsto \mathbf{e}_1 \mapsto \mathbf{0}$ and $[a_{ij}]$ is the matrix of A in the basis $\{\mathbf{e}_i\}$, then the equality $AN = NA$ turns into the system of linear equations $a_{i,j-1} = a_{i+1,j}$ for all $i, j = 1, \ldots, n$, where $a_{i,-1}$ and $a_{n+1,j}$ are understood as zeroes. This implies that A must have constant entries along the diagonals $i + j = const$ (parallel to the main diagonal), and these entries are 0 when $const < 0$. (The last fact, i.e. that A is upper-triangular, is also obvious geometrically since A commuting with N must preserve the complete flag

$$\{0\} \subset \operatorname{Ker} N \subset KerN^2 \subset \cdots \subset \operatorname{Ker} N^{n-1} \subset \mathbb{K}^n$$

associated with the basis $\{\mathbf{e}_i\}$.)

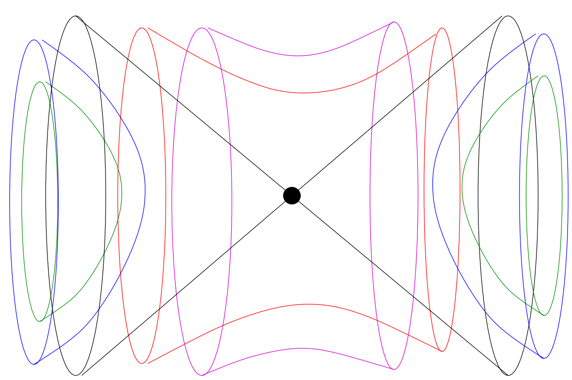
427(d). The characteristic polynomial of the matrix (call it A) is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$. The matrix $A + I = \begin{bmatrix} 4 & 0 & 8 \\ 3 & 0 & -6 \\ -2 & 0 & -4 \end{bmatrix}$ is nilpotent of rank 2, thus has a 1-dimensional kernel, and hence has 1 Jordan cells of sizes 3. Therefore the Jordan canonical form of A is $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

432. A polynomial expression $p(A)$ of $A = CJC^{-1}$ coincides with $Cp(J)C^{-1}$. Thus, it suffices to check that Jordan canonical forms satisfy the Cayley-Hamilton equation. This is immediate for a Jordan cell $J = \lambda_0 I + N$ of size m and eigenvalue λ_0, since its characteristic polynomial is $(\lambda - \lambda_0)^m$ and $(J - \lambda_0 I)^m = N^m = 0$ indeed. For J consisting of r Jordan cells of sizes $m_1, \ldots, m_r$ with eigenvalues $\lambda_1, \ldots, \lambda_r$, the characteristic polynomial is $\prod_i (\lambda - \lambda_i)^{m_i}$ (as J is block-diagonal), and $(J - \lambda_i I)^m_i$ is still block-diagonal, with the ith block turned into 0. Therefore $\prod_i (J - \lambda_i)^m_i$ is the product of (commuting) block-diagonal matrices, where for each diagonal block, one of the factors is 0, and hence the whole product is 0 as expected.

434. The characteristic polynomial of the matrix is $\lambda^2 - (a^2 + bc)$. The (a, b, c)-space is partitioned into the level surfaces (sketched on the next page)

$$a^2 + bc = a^2 + \left(\frac{b+c}{2}\right) - \left(\frac{b-c}{2}\right)^2 = const$$

of the determinant function, which are therefore invariant under similarity transformations. When $const > 0$ (i.e. the determinant $D := -a^2 - bc < 0$), the surfaces are one-sheeted hyperboloids. Their points represent diagonalizable traceless matrices with real eigenvalues $\pm\sqrt{-D}$. When $const < 0$ ($D > 0$), the surfaces are two-sheeted hyperboloids. They correspond to non-real Jordan canonical forms $\begin{bmatrix} i\sqrt{D} & 0 \\ 0 & -i\sqrt{D} \end{bmatrix}$ or, in the real version, the similarity class of $\pm\sqrt{D}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that these normal forms define expansion $\sqrt{D}$ times composed with the 90-degree rotation — counterclockwise for the sign $+$ and clockwise for the sign $-$. They correspond to the two different sheets of the hyperboloid, and are similar under similarity transformations which reverse the orientation of the plane (i.e. similarity transformations which preserve the orientation also preserve each sheet of the hyperboloid). Finally, when $const = 0$ ($D = 0$), the surface is a cone. Its vertex represents the zero matrix, and the two poles (without the vertex) represent the similarity class of the Jordan cells $\pm\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (which belong to different sheets of the cone, and are similar to each other, but only under the transformations reversing the orientation of the plane.)



HW13

438. The characteristic polynomial of the matrix $\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ is $\lambda^2 - 6\lambda - 1$. Its roots are $\lambda_\pm := 3 \pm \sqrt{10}$. The corresponding eigenvectors are $\mathbf{v}_\pm := (c_\pm, 1)^t$ where $3c_\pm + 2 = \lambda_\pm$, i.e. $c_\pm = (1 \pm \sqrt{10})/3$. The initial condition $(13, 13)^t$ is a linear combination $C_+\mathbf{v}_+ + C_1\mathbf{v}_-$ of the eigenvectors (with non-zero coefficients $C_\pm$). The sequence $(x_n, y_n)^t$ is given by the formula $\lambda_+^n C_+\mathbf{v}_+ + \lambda_-^n C_-\mathbf{v}_-$. Since $|\lambda_+| > 1$ and $|\lambda_-| < 1$, the slope x_n/y_n of the radius-vector, when n tends to $\pm\infty$, approaches the slope of the eigenvector $\mathbf{v}_\pm$, i.e. $\lim_{n\to\pm\infty}(x_n/y_n) = (1 \pm \sqrt{10})/3$.

440. The coefficients of the quadratic polynomial with the roots $\lambda_\pm := (3 \pm \sqrt{17})/2$ are found by Vieta's formulas as $\lambda^2 - 3\lambda - 2$. Therefore $a_n := \lambda_+^n + \lambda_-^n$ satisfies the recursion relation $a_{n+1} = 3a_n + 2a_{n-1}$. The initial conditions are $a_0 = 2$, $a_1 = 3$. Thus, both are integers, and a_1 is odd. Therefore by induction, if a_n, a_{n-1} are integers and a_n is odd, then a_{n+1}, a_n are integers, and $a_{n+1} = 3a_n + 2a_{n-1}$ is odd.

441. Let X, Y be two non-commuting 2×2-matrices, $XY \neq YX$. If both have a logarithm, $X = e^A$, $Y = e^B$, then either $e^{A+B} \neq e^A e^B$ or $e^{A+B} \neq e^B e^A$. Following this clue one can find many examples. For instance, the diagonal matrix $X = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} = \exp \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ doesn't commute with the 90° rotation matrix $Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \exp \begin{bmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{bmatrix}$.

444. A Jordan cell J of size m with the eigenvalue λ has $\operatorname{tr} J = m\lambda$. On the other hand, e^J is upper-triangular with e^λ on the diagonal. Therfore $\det e^J = (e^\lambda)^m = e^{m\lambda} = e^{\operatorname{tr} J}$. The same is true for Jordan canonical forms, because they are formed by direct sums of (commuting) Jordan cells J_i and so $\exp(J_1 \oplus \cdots \oplus J_N) = e^{J_1} \cdots e^{J_N}$, and because the determiant is multiplicative, and the trace is additive. Finally, both trace and determiants are invariant under similarity transformations, and therefore the identity $\det e^A = e^{\operatorname{tr} A}$ follows from the Jordan canon0cal form theorem.

445 (h). The roots of $\lambda^4 + 4\lambda^2 + 3 = (\lambda^2 + 3)(\lambda^2 + 1)$ are $\pm i, \pm\sqrt{3}i$. Therefore the general (real) solution has the form $A\,\mathbf{cos}\,t + B\,\mathbf{sin}\,t + C\,\mathbf{cos}\,\sqrt{3}t + D\,\mathbf{sin}\,\sqrt{3}t$, where A, B, C, D are arbitrary real constants. To satisfy the initial condition we must have: $A + C = 1, B + \sqrt{3}D = 0, A + 3C = 0, B + 3\sqrt{3}D = 0$, from which $C = -1/2, A = 3/2, B = D = 0$, i.e. the required solution $x(t) = \frac{3}{2}\,\mathbf{cos}\,t - \frac{1}{2}\,\mathbf{cos}\,\sqrt{3}t$.