SYMPLECTIC GEOMETRY OF FROBENIUS STRUCTURES

ALEXANDER B. GIVENTAL

To Yu. I. Manin on the occasion of his 65th birthday

The concept of a Frobenius manifold was introduced by B. Dubrovin [9] to capture in an axiomatic form the properties of correlators found by physicists (see [8]) in two-dimensional topological field theories "coupled to gravity at the tree level". The purpose of these notes is to reiterate and expand the viewpoint, outlined in the paper [7] of T. Coates and the author, which recasts this concept in terms of linear symplectic geometry and exposes the role of the twisted loop group $\mathcal{L}^{(2)}GL_N$ of hidden symmetries.

We try to keep the text introductory and non-technical. In particular, we supply details of some simple results from the axiomatic theory, including a several-line proof of the genus 0 Virasoro constraints, but merely quote and refer to the literature for a number of less trivial applications, such as the quantum Hirzebruch–Riemann–Roch theorem in the theory of cobordism-valued Gromov–Witten invariants. The latter is our joint work in progress with Tom Coates, and we would like to thank him for numerous discussions of the subject.

The author is also thankful to Claus Hertling, Yuri Manin and Matilde Marcolli for the invitation to the workshop on Frobenius structures held at MPIM Bonn in Summer 2002.

Gravitational descendents. In Witten's formulation of topological gravity [44] one is concerned with certain "correlators" called *gravitational descendents*. The totality of genus 0 gravitational descendents is organized as the set of Taylor coefficients of a single formal function $\mathcal{F}$ of a sequence of vector variables $\mathbf{t} = (t_0, t_1, t_2, ...)$. The vectors t_i are elements of a finite-dimensional vector space[1] which we will denote H. One assumes that H is equipped with a symmetric non-degenerate bilinear form $(\cdot, \cdot)$ and with a distinguished non-zero element $\mathbf{1}$. The axioms are formulated as certain partial differential equations on $\mathcal{F}$: an infinite set of *Topological Recursion Relations* (TRR), the *String Equation* (SE) and the *Dilaton Equation* (DE). To state the axioms using the following coordinate notation: introduce a basis $\{\phi_\alpha\}$ in H such that $\phi_1 = \mathbf{1}$, put $t_k = \sum t_k^\alpha \phi_\alpha$, $(\phi_\alpha, \phi_\beta) = g_{\alpha\beta}$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ and write $\partial_{\alpha,k}$ for the partial derivative $\partial/\partial t_k^\alpha$. Then the axioms read:

$$(\text{DE}) \qquad \partial_{1,1}\mathcal{F}(\mathbf{t}) = \sum_{n=0}^{\infty} \sum_{\nu} t_n^\nu \partial_{\nu,n}\mathcal{F}(\mathbf{t}) \; - \; 2\mathcal{F}(\mathbf{t}),$$

[1]or super-space, but we will systematically ignore the signs that may come out of this possibility.

(SE) $$\partial_{1,0}\mathcal{F}\left(\mathbf{t}\right) = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty}\sum_{\nu} t_{n+1}^{\nu}\partial_{\nu,n}\mathcal{F}\left(\mathbf{t}\right),$$

(TRR) $$\partial_{\alpha,k+1}\partial_{\beta,l}\partial_{\gamma,m}\mathcal{F}\left(\mathbf{t}\right) = \sum_{\mu\nu}\partial_{\alpha,k}\partial_{\mu,0}\mathcal{F}\left(\mathbf{t}\right)\,g^{\mu\nu}\,\partial_{\nu,0}\partial_{\beta,l}\partial_{\gamma,m}\mathcal{F}\left(\mathbf{t}\right)$$

for all α, β, γ and any $k, l, m \geq 0$. We will see soon that the system DE+SE+TRR admits a transparent geometrical interpretation.

Let $\mathcal{H}$ denote the *loop space* $H((z^{-1}))$ consisting of Laurent series in $1/z$ with vector coefficients. Introduce the symplectic form Ω on $\mathcal{H}$:

$$\Omega(\mathbf{f}, \mathbf{g}) = \frac{1}{2\pi i}\oint\,(\mathbf{f}(-z), \mathbf{g}(z))\,dz.$$

The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ by the Lagrangian subspaces $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$ provides a symplectic identification of $(\mathcal{H}, \Omega)$ with the cotangent bundle $T^*\mathcal{H}_+$. We encode the domain variables $\mathbf{t} = (t_0, t_1, t_2, ...)$ of the function $\mathcal{F}$ in the coefficients of the vector polynomial

$$\mathbf{q}(z) = -z + t_0 + t_1 z + t_2 z^2 + ... \qquad (\text{where } -z := -\mathbf{1}z)\ .$$

Thus the formal function $\mathcal{F}(\mathbf{t})$ near $\mathbf{t} = 0$ becomes a formal function on the space $\mathcal{H}_+$ near the shifted origin $\mathbf{q} = -z$. This convention is called the *dilaton shift*. Denote by $\mathcal{L}$ the graph of the differential $d\mathcal{F}$:

$$\mathcal{L} = \{(\mathbf{p}, \mathbf{q}) \in T^*\mathcal{H}_+ :\ \mathbf{p} = d_{\mathbf{q}}\mathcal{F}\}.$$

This is a formal germ at $\mathbf{q} = -z$ of a Lagrangian section of the cotangent bundle $T^*\mathcal{H}_+$ and can therefore be considered as a formal germ of a Lagrangian submanifold in the symplectic loop space $(\mathcal{H}, \Omega)$.

Theorem 1. *The function $\mathcal{F}$ satisfies TRR, SE and DE if and only if the corresponding Lagrangian submanifold $\mathcal{L} \subset \mathcal{H}$ has the following property:*

($\star$)   *$\mathcal{L}$ is a Lagrangian cone with the vertex at the origin and such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along zL.*

More precisely, the last condition is interpreted as follows. First, it shows that the tangent spaces $L = T_{\mathbf{f}}\mathcal{L}$ satisfy $zL \subset L$ (and therefore $\dim L/zL = \dim \mathcal{H}_+/z\mathcal{H}_+ = \dim H$). Second, it implies that $zL \subset \mathcal{L}$. Third, the same L is the tangent space to $\mathcal{L}$ not only along the line of $\mathbf{f}$ but also at all smooth points in zL. This is meant to include $\mathbf{f}$, i.e. $L = T_{\mathbf{f}}\mathcal{L}$ implies $\mathbf{f} \in zL$.

The tangent spaces L vary therefore in a $\dim H$-parametric family. In particular they generate a *variation of semi-infinite Hodge structures* in the sense of S. Barannikov [1], i.e. a family of semi-infinite flags $\cdots zL \subset L \subset z^{-1}L \cdots$ satisfying the Griffiths integrability condition.

Thus another way to phrase the property ($\star$) is to say that
$\mathcal{L}$ is a cone ruled by the isotropic subspaces zL varying in a $\dim H$-parametric family with the tangent spaces along zL equal to the same Lagrangian space L.

Finally, the property should be understood in the sense of formal geometry near a point $\mathbf{f}_0$ different from the origin. In particular, the fact that cones develop singularities at the origin does not contradict the fact that $\mathcal{L}$ has Lagrangian tangent spaces as a section of $T^*\mathcal{H}_+$ near $\mathbf{q} = -z$.

Proof. Let us begin with the cone $\mathcal{L}$ satisfying $(\star)$. Using the polarization $\mathcal{H} = T^*\mathcal{H}_+$ we recover $\mathcal{F}$ as the restriction of the quadratic form $\mathbf{p}(\mathbf{q})/2$ (evaluation of the covector on a vector) to the cone $\mathcal{L}$ parameterized by $\mathcal{H}_+$ as a section of the cotangent bundle. The resulting function $\mathcal{F}$ is homogeneous of degree 2 (since $\mathcal{L}$ is a cone) which is exactly what DE expresses after the dilaton shift.

Multiplication by z^{-1} in $\mathcal{H}$ is anti-symmetric with respect to the symplectic form Ω and thus defines a linear hamiltonian vector field $\mathbf{f} \mapsto z^{-1}\mathbf{f}$ corresponding to the quadratic hamiltonian $\Omega(z^{-1}\mathbf{f}, \mathbf{f})/2$. The vector field is tangent to $\mathcal{L}$ since $z^{-1}\mathbf{f}$ is contained in the tangent space L at the point $\mathbf{f} \in zL$, The cone $\mathcal{L}$ is therefore situated in the zero level of the quadratic hamiltonian. It is straightforward to check that this condition written as the Hamilton–Jacobi equation for the generating function $\mathcal{F}$ coincides with the string equation.

Next, the property of $\mathcal{L}$ to have constant tangent spaces L along subspaces zL of codimension $\dim H$ shows that the quadratic differentials $d_{\mathbf{t}}^2\mathcal{F}$ depend only on $\dim H$ variables. In particular this applies to $\tau^\delta := \sum g^{\delta\mu}\partial_{\mu,0}\partial_{1,0}\mathcal{F}$. Differentiating the string equation in t_0^μ we find

$$(1) \qquad \tau^\delta = t_0^\delta + \sum_{n,\mu,\nu} g^{\delta\mu}\, t_{n+1}^\nu\, \partial_{\nu,n}\partial_{\mu,0}\mathcal{F}.$$

In particular $\tau^\delta(\mathbf{t})|_{t_1=t_2=\ldots=0} = t_0^\delta$ are independent and thus τ^δ can be taken in the role of the $\dim H$ variables.

Note that the tangent spaces L are transverse to $\mathcal{H}_-$ and so the spaces zL are transverse to $z\mathcal{H}_-$. Therefore the projection of zL to the base $\mathcal{H}_+$ along $\mathcal{H}_-$ is transverse to the slice $q_1 = -\mathbf{1}, q_2 = q_3 = \ldots = 0$, and the intersection with the slice has $q_0 = \tau$. Making use of the correlator notation

$$\langle \phi_\alpha\psi^k, \psi_\beta\psi^l, \ldots, \phi_\gamma\psi^m \rangle(\tau) := (\partial_{\alpha,k}\partial_{\beta,l}\ldots\partial_{\gamma,m}\mathcal{F})|\, t_0 = \tau, t_1 = t_2 = \ldots = 0,$$

we can write

$$(2) \qquad \partial_{\beta,l}\partial_{\gamma,m}\mathcal{F}(\mathbf{t}) = \langle \phi_\beta\psi^l, \phi_\gamma\psi^m \rangle(\tau(\mathbf{t})).$$

Consider (1) as an implicit equation for $\tau(\mathbf{t})$:

$$(3) \qquad G^\delta(\tau, \mathbf{t}) := \tau^\delta - t_0^\delta - \sum_{n,\mu,\nu} g^{\delta\mu}\, t_{n+1}^\nu\, \langle \phi_\nu\psi^l, \phi_\mu \rangle(\tau) = 0.$$

Using the rules of implicit differentiation we conclude that the matrix $(\partial_{\lambda,0}\tau^\delta)$ is inverse to $(\partial G^\delta/\partial\tau^\lambda)$ at $\tau = \tau(\mathbf{t})$. Using this together with implicit differentiation once again, we find

$$\partial_{\alpha,k+1}\tau^\delta = \sum_{\mu,\nu} \partial_{\nu,0}\tau^\delta\, g^{\mu\nu}\, \langle \phi_\alpha\psi^k, \phi_\mu \rangle(\tau(\mathbf{t})).$$

This is a special case of TRR whose general form is obtained now by differentiation of (2):

$$
\begin{aligned}
\partial_{\alpha,k+1}\partial_{\beta,l}\partial_{\gamma,m}\mathcal{F} &= \sum_\delta \partial_{\alpha,k+1}\tau^\delta\, \langle \phi_\delta, \phi_\beta\psi^l, \phi_\gamma\psi^m \rangle \\
(4) \qquad &= \sum_{\delta,\mu,\nu} \langle \phi_\alpha\psi^k, \phi_\mu \rangle\, g^{\mu\nu}\, \partial_{\nu,0}\tau^\delta\, \langle \phi_\delta, \phi_\beta\psi^l, \phi_\gamma\psi^m \rangle \\
&= \sum_{\mu,\nu} \langle \phi_\alpha\psi^k, \phi_\mu \rangle\, g^{\mu\nu}\, \partial_{\nu,0}\partial_{\beta,l}\partial_{\gamma,m}\mathcal{F}.
\end{aligned}
$$

In the opposite direction, suppose that $\mathcal{F}$ satisfies TRR, SE and DE. It is straightforward to see that given $\langle \phi_\beta \psi^l, \phi_\gamma \psi^l \rangle(\tau)$, the TRR uniquely determines all the other correlators with at least two seats.

We use (3) to implicitly define the (formal) map $\mathbf{t} \mapsto \tau(\mathbf{t})$ and try the right hand side of (2) in the role of $\partial_{\beta,l} \partial_{\gamma,m} \mathcal{F}$. Repeating the above derivation (4) we conclude that TRR is satisfied and hence, by the uniqueness, (2) is indeed true.

We see therefore that the tangent spaces to the graph $\mathcal{L}$ of $d\mathcal{F}$ are constant along the fibers of the map $\mathbf{t} \mapsto \tau$. [These fibers are affine subspaces (of codimension $\dim H$) since G is linear inhomogeneous in $\mathbf{t}$ and $\tau(t_0, 0, 0, ...) = t_0$. Taking $\mathbf{t} = (0, 1, 0, 0, ...)$ in G we find $G^\delta = \tau^\delta - \sum g^{\delta,\mu} \langle \phi_\mu, \mathbf{1} \rangle(\tau)$. This equals 0 for all τ because $\langle \phi_\mu, \mathbf{1} \rangle(\tau) = (\phi_\mu, \tau)$ as it follows from the string equation. Thus the fibers of the map $\mathbf{t} \mapsto \tau$ become linear subspaces after the dilaton shift.]

Due to the dilaton equation, $\mathcal{F}$ as a function on $\mathcal{H}_+$ is homogeneous of degree 2, and hence $\mathcal{L}$ is homogeneous of degree 1. Thus $\mathcal{L}$ is a cone, so the tangent spaces $L = T_{\mathbf{f}}\mathcal{L}$ contain the application points $\mathbf{f}$. Recalling that the string equation expresses the tangency of $z^{-1}\mathbf{f}$ to $\mathcal{L}$ at the points $\mathbf{f} \in \mathcal{L}$ we see that in fact $\mathbf{f} \in L \cap zL$. This is true for all those $\mathbf{f}$ where the tangent space is L. Therefore the projection of zL to $\mathcal{H}_+$ along $\mathcal{H}_-$ contains one of the fibers of the map $\mathbf{t} \mapsto \tau$. In fact the projection coincides with the fiber since both have codimension $\dim H$ in $\mathcal{H}_+$. For the space zL itself this means that it coincides with the set of points $\mathbf{f} \in \mathcal{L}$ where $T_{\mathbf{f}}\mathcal{L} = L$. In particular $zL \subset L$. $\square$

The property $(\star)$ is formulated in terms of the symplectic structure Ω and the operator of multiplication by z but it does not depend on the choice of the polarization. This shows that the system DE+SE+TRR has a huge group of hidden symmetries. Let $\mathcal{L}^{(2)}GL(H)$ denote the *twisted loop group* which consists of $\text{End}(H)$-valued formal Laurent series M in the indeterminate z^{-1} satisfying $M^*(-z)M(z) = \mathbf{1}$. Here $*$ denotes the adjoint with respect to $(\cdot, \cdot)$. The following corollary shows how the action of the twisted loop group can yield new functions $\mathcal{F}$ which satisfy the system DE+SE+TRR whenever the original one does.

Corollary. *The action of the twisted loop group preserves the class of the Lagrangian cones $\mathcal{L}$ satisfying $(\star)$.*

Frobenius structures. By a result of S. Barannikov [1], the set of tangent spaces L to the cone $\mathcal{L}$ satisfying $(\star)$ carries a canonical Frobenius structure. We outline Barannikov's construction.

Consider the intersection of the cone $\mathcal{L}$ with the affine space $-z + z\mathcal{H}_-$. The intersection is parameterized by $\tau \in H$ via its projection to $-z + H$ along $\mathcal{H}_-$ and can be considered as the graph of a function from H to $\mathcal{H}$ called the *J-function*:

$$\tau \mapsto J(-z, \tau) = -z + \tau + \sum_{k>0} J_k(\tau)(-z)^{-k}.$$

The derivatives $\partial J/\partial \tau^\delta = \phi_\delta + ...$ form a basis in $L \cap z\mathcal{H}_-$ which is a complement to zL in L. Since $z\partial J/\partial \tau^\delta \in zL \subset \mathcal{L}$, the 2nd derivatives $z\partial^2 J/\partial \tau^\delta \partial \tau^\lambda$ are in L again. On the other hand, these 2nd derivatives are in $z\mathcal{H}_-$ as well. Thus they are uniquely representable via the basis as $\sum A_{\delta\lambda}^\mu \partial J/\partial \tau^\mu$ with the coefficients $A_{\delta\lambda}^\mu(\tau)$ independent of z. Thus $\partial J/\partial \tau^d$ form a fundamental solution of the pencil of flat

connections depending linearly on z^{-1}:

$$(5) \qquad z\frac{\partial}{\partial\tau^\lambda}\frac{\partial J}{\partial\tau^\delta} = \sum_\mu A^\mu_{\delta\lambda}(\tau)\frac{\partial J}{\partial\tau^\mu}.$$

Obviously $A^\mu_{\delta\lambda} = A^\mu_{\lambda\delta}$. Moreover, the string flow $\mathbf{f} \mapsto \exp(u/z)\mathbf{f}$ preserves $-z+z\mathcal{H}_-$ and projects along $\mathcal{H}_-$ to $-z + \tau \mapsto -z + \tau - u\mathbf{1}$. This implies that $z\partial J/\partial\tau^1 = J$ and respectively $(A^\mu_{1\lambda})$ is the identity matrix. Furthermore, it is not hard to derive now that the multiplications on the tangent spaces $T_\tau H = H$ given by

$$\phi_\delta \bullet \phi_\lambda = \sum_\mu A^\mu_{\delta\lambda}(\tau)\phi_\mu$$

define associative commutative algebra structures with the unit $\mathbf{1}$ which are Frobenius with respect to the inner product on H (i.e. $(a \bullet b, c) = (a, b \bullet c)$) and satisfy the integrability condition imposed by (5). This is what a Frobenius structure is (see for instance [39]).[2]

Conversely, given a Frobenius structure one recovers a J-function by looking for a fundamental solution to the system (*cf.* (5))

$$z\frac{\partial}{\partial\tau^\lambda}S = \phi_\lambda \bullet S$$

in the form of an operator-valued series $S = \mathbf{1} + S_1(\tau)z^{-1} + S_2(\tau)z^{-2} + ...$ satisfying $S^*(-z)S(z) = \mathbf{1}$. Such a solution always exists and yields the corresponding J-function $J^\delta(z,\tau) = z[S^*(z,\tau)]^\delta_1$ and the Lagrangian cone $\mathcal{L}$ enveloping the family of Lagrangian spaces $L = S^{-1}(z,\tau)\mathcal{H}_+$ and satisfies the condition $(\star)$. A choice of the fundamental solution S is called in [25] a *calibration* of the Frobenius structure. The calibration is unique up to the right multiplication $S \mapsto SM$ by elements $M = \mathbf{1} + M_1 z^{-1} + M_2 z^{-2} + ...$ of the "lower-triangular" subgroup in the twisted loop group. Thus the action of this subgroup on our class of cones $\mathcal{L}$ changes calibrations but does not change Frobenius structures (while more general elements of $\mathcal{L}^{(2)}GL(H)$, generally speaking, change Frobenius structures as well).

As an example, consider the translation invariant Frobenius manifold defined by a given Frobenius algebra structure $\circ$ on H. The system (5) has constant coefficients and allows for the following obvious solution

$$J(z,\tau) = z\ e^{(\tau\circ)/z} = \sum_{k\geq 0} z^{1-k}\frac{\tau^{\circ k}}{k!}.$$

The corresponding cone is $\mathcal{L} = \{\ \mathbf{e}^{(\tau\circ)/z}\mathbf{q}(z),\ \mathbf{q} \in H[z]\ \}$. Infinitesimal automorphisms of $\mathcal{L}$ in the twisted loop Lie algebra contains the subalgebra

$$z^{-1}(a_0\circ) + z(a_1\circ) + z^3(a_2\circ) + ...,\ a_i \in H$$

and actually coincides with it when the algebra $(H, \circ)$ is semisimple.

Quantum cohomology. In Gromov – Witten theory, the genus 0 descendent potential $\mathcal{F}$ of a compact (almost) Kähler manifold X is defined in terms on intersection theory on moduli spaces $X_{0,n,d}$ of degree $d \in H_2(X, \mathbb{Z})$ stable maps $\Sigma \to X$

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[2]To match this with Dubrovin's original definition [9] one can use the generating function $\mathcal{F}$ to show that in fact $A^\mu_{\delta\lambda} = \sum g^{\mu\nu}\ \langle\phi_\nu, \phi_\delta, \phi_\lambda\rangle(\tau)$ and therefore $\mathcal{F}(\tau, 0, 0, ...)$ satisfies the WDVV-equation.

of genus 0 complex curves with n marked points [34]. Namely

$$(6) \qquad \mathcal{F} = \sum_{d,n} \frac{Q^d}{n!} \int_{[X_{0,n,d}]} \prod_{i=1}^{n} (\sum_{k=0}^{\infty} \mathrm{ev}_i^*(t_k)\psi_i^k).$$

Here $[X_{0,n,d}]$ is the virtual fundamental cycle [38], $t_k \in H^*(X, \mathbb{Q})$ are cohomology classes of the target space X to be pulled back to the moduli spaces by the evaluation maps $\mathrm{ev}_i : X_{0,n,d} \to X$ at the i-th marked points, and ψ_i is the 1-st Chern class of the *universal cotangent line bundle* L_i. The fiber of L_i is defined as the cotangent line $T_{\sigma_i}^* \Sigma$ to the curve at the i-th marked point.

Thus the cohomology of the target space equipped with the Poincaré intersection pairing

$$g_{\mu\nu} = \int_X \phi_\mu \phi_\nu$$

plays the role of the inner product super-space H. Yet the situation here does not quite fit our axiomatic set-up since an appropriate *Novikov ring* $\mathbb{Q}[[Q]]$ takes the place of the ground field. Moreover, one often has to consider the Lagrangian section $\mathcal{L} = \mathrm{graph} \, d\mathcal{F}$ in the completion of the space $\mathcal{H}$ consisting of the series $\sum f_k z^k$ infinite in both directions but such that $f_k \to 0$ as $k \to +\infty$ in the Q-adic topology of $H = H^*(X, \mathbb{Q}[[Q]])$. In what follows we will allow this and similar generalizations of the framework without special notice.

The genus 0 descendent potential is known to satisfy the axioms DE, SE and TRR and therefore defines on H a Frobenius structure with the multiplication $\bullet$ known as *quantum cup-product*. This Frobenius structure comes therefore equipped with the canonical calibration S and the J-function:

$$(7) \qquad J(z,\tau) = z + \tau + \sum_{d,n} \frac{Q^d}{n!} \, (\mathrm{ev}_{n+1})_* \big[\frac{\prod_{i=1}^{n} \mathrm{ev}_i^*(\tau)}{z - \psi_{n+1}}\big],$$

where $(\mathrm{ev}_{n+1})_*$ is the push-forward along the evaluation map $\mathrm{ev}_{n+1} : X_{0,n+1,d} \to X$. The present description of this expression as the defining section of the whole cone $\mathcal{L}$ seems more satisfactory than the *ad hoc* construction in the literature on mirror formulas.

Below we briefly recall several results from [7] exploiting the language of Lagrangian cones in the context of genus 0 Gromov – Witten theory.

Gravitational ancestors. Forgetting the maps $\Sigma \to X$ (and maybe several last marked points) and stabilizing the remaining marked curves one defines the *contraction maps* $\mathrm{ct} : X_{g,m+n,d} \to \overline{\mathcal{M}}_{g,m}$ of moduli spaces of stable maps to the Deligne – Mumford spaces. The latter carry their own universal cotangent line bundles and their 1-st Chern classes whose pull-backs to $X_{g,m+n,d}$ will be denoted $\bar{L}_i$ and $\bar{\psi}_i$ respectively. We call *ancestors* the Gromov – Witten invariants which "mistakenly" use the classes $\bar{\psi}$ instead of ψ. A theorem of Kontsevich – Manin (see Theorem 2.2 in [35]) expresses descendents via ancestors and vice versa. [3] It was E. Getzler [17, 19] who emphasized a primary role of ancestors in the theory of topological recursion relations. The following theorem interprets their results geometrically in the case $g = 0$.

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[3] The underlying relationship between ψ and $\bar{\psi}$ is described also in [17], Proposition 5.

Introduce the genus 0 *ancestor potential* $\bar{\mathcal{F}}_\tau$ depending on the parameter $\tau \in H$:

$$\bar{\mathcal{F}}_\tau = \sum_{d,m,n} \frac{Q^d}{m!n!} \int_{[X_{0,n,d}]} \prod_{i=1}^{m} (\sum_{k=0}^{\infty} \mathrm{ev}_i^*(\bar{t}_k)\bar{\psi}_i^k) \prod_{j=m+1}^{m+n} \mathrm{ev}_j^*(\tau).$$

Let $\mathcal{L}_\tau$ denotes the Lagrangian section in $\mathcal{H} = T^*\mathcal{H}_+$ representing the graph of $d\bar{\mathcal{F}}_\tau$ subject, as usual, to the dilaton shift $\mathbf{q}(z) = -z + \sum \bar{t}_k z^k$.

Theorem 2 (see Appendix 2 in [7]). *We have $\mathcal{L}_\tau = S(\tau)\mathcal{L}$, where $\mathcal{L}$ represents the descendent potential, and $S(\tau) \in \mathcal{L}^{(2)}GL(H)$ is the corresponding calibration.*

The ancestor potential $\bar{\mathcal{F}}_0$ has zero 2-jet when $\bar{t}_0 = 0$ since $\dim \overline{\mathcal{M}}_{0,2+m} = m - 1 < m$. Therefore $\mathcal{L}_\tau$ is tangent to $\mathcal{H}_+$ along $z\mathcal{H}_+$. Thus Theorem 2 implies directly (i.e. by-passing Theorem 1) that the graph $\mathcal{L}$ of $d\mathcal{F}$ satisfies $(\star)$.

Corollary. *The sections $\mathcal{L}_\tau$ are themselves Lagrangian cones satisfying the condition $(\star)$ (and isomorphic to each other and to $\mathcal{L}$).*

The Frobenius structures defined by the cones $\mathcal{L}_\tau$ are isomorphic to that for $\mathcal{L}$ since the calibration S is "lower-triangular".

On the other hand, Theorem 2 tells us how to define the ancestor potentials $\bar{\mathcal{F}}_\tau$ in the axiomatic theory: restrict the quadratic form $\mathbf{p}(\mathbf{q})/2$ to $S(\tau)\mathcal{L}$. In particular, the genus 0 ancestor potentials $\bar{\mathcal{F}}_\tau$ satisfy SE, DE and TRR (which agrees with Theorem C in [43]).

Moreover, let us call a function $\mathcal{G}(\bar{t}_0, \bar{t}_1, ...)$ *tame* if

$$(8) \qquad \frac{\partial}{\partial \bar{t}_{k_1}^{\alpha_1}} \cdots \frac{\partial}{\partial \bar{t}_{k_r}^{\alpha_r}} \mathcal{G} \mid_{\bar{\mathbf{t}}=0} = 0 \quad \text{whenever} \quad k_1 + ... + k_r > r - 3.$$

The ancestor potentials in Gromov – Witten theory are tame on the dimensional grounds: $\dim \overline{\mathcal{M}}_{0,r} = r - 3$. The following result shows that in the axiomatic theory there is no need to add this condition as a new axiom.

Proposition. *The ancestor potentials $\bar{\mathcal{F}}_\tau$ (defined in the axiomatic theory as the generating functions for the Lagrangian cones $\mathcal{L}_\tau := S(\tau)\mathcal{L}$) are tame.*

Proof. The ancestor potentials themselves satisfy DE+SE+TRR but in addition have zero 2-jet when $\bar{t}_1 = \bar{t}_2 = ... = 0$. Let us apply the derivation $\partial^r/\partial \bar{t}_{n_1}^{\alpha_1}...\partial \bar{t}_{n_r}^{\alpha_r}$ to the identity TRR for $\bar{F}_\tau$ and assume that (8) is satisfied for the terms on the right hand side. If at least one of these terms is non-zero at $\bar{\mathbf{t}} = 0$, then we must have $k + l + m + n_1 + ... + n_r \le r - 1$. Thus (8) is satisfied for the left hand side. This allows us to derive (8) by induction on $k_1 + ... + k_r$. $\square$

Apparently, a more general result applicable to higher genus theory is contained in [19].

Twisted Gromov – Witten invariants. Let E be a holomorphic vector bundle over a Kähler target space X. One can pull it back to a moduli space $X_{g,n+1,d}$ by the evaluation map $\mathrm{ev}_{n+1} : X_{g,n+1,d} \to X$ and then push the result forward to $X_{g,n,d}$ by the map $\mathrm{ft}_{n+1} : X_{g,n+1,d} \to X_{g,n,d}$ forgetting the $n+1$-st marked point. This way one gets an element in $K^0(X_{g,n,d})$:

$E_{g,n,d} = $ a virtual bundle with the fibers $H^0(\Sigma; f^*E) \ominus H^1(\Sigma; f^*E)$.

Fixing an invertible multiplicative characteristic class $S(\cdot) = e^{\sum s_k ch_k(\cdot)}$ of complex vector bundles, one defines *twisted* GW-invariants by systematically using intersection indices against the *twisted* virtual fundamental cycle $S(E_{g,n,d}) \cap [X_{g,n,d}]$.

When E is a negative line bundle, and S is the inverse total Chern class, the twisted theory represents GW-invariants of the total space of the bundle. When E is a positive line bundle, and S is the Euler class, the twisted genus 0 theory is very close to the GW-theory of the hypersurface defined by a section of the line bundle. The following "Quantum Riemann – Roch theorem" describes the twisted version $\mathcal{F}^{tw}$ of the genus 0 descendent potential (6) via the untwisted one.

Given E and S, represent the the twisted genus 0 descendent potential by the Lagrangian submanifold $\mathcal{L}^{tw} \subset \mathcal{H}$ obtained from the graph of $d\mathcal{F}^{tw}$ using the dilaton shift convention $\mathbf{q}(z) = \sqrt{S(E)}(\mathbf{t}(z) - z)$.

Theorem 3 (see [7]). *The Lagrangian submanifold $\mathcal{L}^{tw}$ is obtained from the Lagrangian cone $\mathcal{L}$ of the untwisted theory by a linear symplectic transformation:*

$$\mathcal{L}^{tw} = \triangle\mathcal{L}, \quad \triangle \sim \sqrt{S(E)} \prod_{m=1}^{\infty} S(E \otimes L^{-m})$$

where L is a line bundle with $c_1(L) = z$.

More precisely, put $s(x) = \sum s_k x^k / k!$ and let x_i be the Chern roots of E, so that $S(E) = \exp \sum_i s(x_i)$. Then the expression

$$\ln\{S(E)^{1/2} \prod_{m=1}^{\infty} S(E \otimes L^{-m})\} = \sum_i [\frac{s(x_i)}{2} + \sum_{m=1}^{\infty} s(x_i - mz)]$$

has the asymptotical expansion

$$\frac{1}{2} \sum_i \frac{1 + e^{-z\partial/\partial x_i}}{1 - e^{-z\partial/\partial x_i}} \, s(x_i) = \sum_{0 \leq m} \sum_{0 \leq l \leq \dim_{\mathbb{C}} X} s_{2m-1+l} \frac{B_{2m}}{(2m)!} ch_l(E) z^{2m-1},$$

where B_{2m} are the Bernoulli numbers, and $ch_l(E)$ are understood as operators of multiplication in the cohomology algebra H of the target space X. The operator $\ln \triangle$ is the infinitesimal symplectic transformation defined by this series.

When expressed in the original terms of the generating functions, the relationship stated in the theorem would have the form

$$\mathcal{F}^{tw}(\mathbf{q}) = \text{ critical value in } \mathbf{q}' \text{ of } \mathcal{G}(\mathbf{q}, \mathbf{q}') + \mathcal{F}(\mathbf{q}'),$$

where $\mathcal{G}$ is the quadratic generating function of the above linear symplectic transformation. The operation of computing the critical value is complicated, and it seems not obvious if the new function $\mathcal{F}^{tw}$ satisfies, say, the TRR. In fact it is not hard to see directly that the twisted genus 0 GW-theory satisfies the same axioms DE, SE and TRR. Then another question arises: are these equations for $\mathcal{F}^{tw}$ formal consequences of those for $\mathcal{F}$ or they carry some additional information about the untwisted theory? The use of Lagrangian cones makes the answer obvious: since the linear transformation in the theorem belongs to the group $\mathcal{L}^{(2)}GL(H)$, the condition $(\star)$ for $\mathcal{L}^{tw}$ is formally equivalent to that for $\mathcal{L}$. This elementary example illustrates possible advantages of the geometrical viewpoint.

Quantum Lefschetz Theorem. The following result is a formal (but combinatorially non-trivial) consequence of Theorem 3.

Note that the germ of a Lagrangian cone $\mathcal{L}$ satisfying $(\star)$ is determined by a $\dim H$-dimensional submanifold (at least when it is sufficiently close to the J-function). We begin with the J-function $J(-z, \tau)$ representing the Lagrangian cone $\mathcal{L}$ of the untwisted theory on X and generate a new Lagrangian cone $\mathcal{L}'$ satisfying

$(\star)$ by using the following function $I(-z, \tau)$ in the role of "a $\dim H$-dimensional submanifold":

$$I(z, \tau) \sim \int_0^\infty e^{x/z} J(z, \tau + \rho \ln x) \, dx \; \Big/ \; \int_0^\infty e^{(x - \rho \ln x)/z} dx,$$

where $\rho = \lambda + c_1(E)$ is the total Chern class of a complex line bundle E over X, and the integrals should be understood in the sense of stationary phase asymptotics as $z \to 0$.

Theorem 4 (see [7]). *The Lagrangian cone $\mathcal{L}'$ determined by the function I coincides with the cone $\mathcal{L}^{tw}$ of the genus 0 Gromov – Witten theory on X twisted by the line bundle E and by the total Chern class*

$$S(\cdot) = \lambda^{\dim(\cdot)} + c_1(\cdot) \lambda^{\dim(\cdot) - 1} + ... + c_{\dim(\cdot)}(\cdot).$$

This is an abstract formulation of a very general result which in the limit $\lambda \to 0$ allows one to compute genus 0 GW-invariants of a hypersurface in X defined by a section of E in a form generalizing Quantum Lefschetz Theorems of B. Kim, A. Bertram, Y.-P. Lee and A. Gathmann [2, 3, 36, 16]. Namely, Theorem 4 applies beyond "small" quantum cohomology theory, to general type complete intersections as well, and also offers a new insight on the nature of *mirror maps* as a very special case of *Birkhoff factorization* in loop groups. In particular, given the J-function of a toric Fano manifold X, the theorem yields the mirror formulas of [22] for the J-function of a Fano or Calabi – Yau complete intersection in X (including the celebrated example of quintic three-folds in $\mathbb{C}P^4$).[4]

Singularity theory. Although not named so, Frobenius structures first emerged around 1980 from K. Saito's theory of primitive forms in singularity theory (see [29] for a modern exposition).

Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function (for simplicity — weighted homogeneous) at an isolated critical point of multiplicity N. Let $F(x, \tau)$ be a miniversal deformation of f, i.e. a family of functions in $\mathbf{x} \in \mathbb{C}^m$ depending on the N-dimensional parameter $\tau \in \mathcal{T}$ and such that $F(x, 0) = f(x)$ and $\partial F / \partial \tau^\alpha |_{\tau = 0}$, $\alpha = 1, ..., N$, represent a basis in the local algebra H of the critical point:

$$H = \mathbb{C}[x]/(f_{x_1}, ..., f_{x_m}).$$

The tangent spaces $T_\tau \mathcal{T}$ are canonically identified with the algebras $\mathbb{C}[x]/(F_x)$ of functions on the critical sets. To make the algebras $T_\tau \mathcal{T}$ Frobenius, one picks a holomorphic volume form ω on $\mathbb{C}^m$ (possibly depending on τ) and introduces the *residue pairing* on $T_\tau \mathcal{T}$ via

$$(a, b) = \left(\frac{1}{2\pi i}\right)^m \oint ... \oint a(y) \, b(y) \, \frac{dy_1 \wedge ... \wedge dy_m}{F_{y_1} \, ... \, F_{y_m}}.$$

(We assume here that y is a unimodular coordinate system on $\mathbb{C}^m$, i.e. $\omega = dy_1 \wedge ... \wedge dy_m$.) To make $\mathcal{T}$ a Frobenius manifold one takes the volume form ω to be

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[4]The initial proof [21, 22] of the mirror formulas for toric complete intersections was based on fixed point localization while the new approaches [36, 16, 7] seem to be free of this restrictive technique. It is worth mentioning however that these approaches require the J-function of the ambient toric manifold as an input. Ironically, the only currently available method of computing J-functions of general Fano toric manifolds is to specialize the result of [22] based on fixed point localization to toric complete intersections of codimension 0.

primitive, i.e. satisfying very special conditions which in particular guarantee that the residue metric is flat.

Many attributes of abstract Frobenius structures make more sense when interpreted in terms of singularity theory. For example, the system (5) is satisfied by the complex oscillating integrals

$$\mathcal{J}_{\mathfrak{B}}(\tau) = (-2\pi z)^{-m/2} \int_{\mathfrak{B}} e^{F(x,\tau)/z}\omega$$

written in flat coordinates τ of the residue metric on $\mathcal{T}$.

The oscillating integrals also satisfy the following weighted homogeneity condition:

$$(9) \qquad (z\frac{\partial}{\partial z} + \sum(\deg \tau^{\lambda})\tau^{\lambda}\frac{\partial}{\partial \tau^{\lambda}})\ z\frac{\partial}{\partial \tau^{\delta}}\mathcal{J} = -\mu_{\delta}\ z\frac{\partial}{\partial \tau^{\delta}}\mathcal{J},$$

where $\mu_{\delta} = \deg(\partial_{\delta}F) + \deg(\omega) - m/2,\ \delta = 1, ..., N,$ is the *spectrum* of the singularity symmetric about 0. The pencil (5) of flat connections over $\mathcal{T}$ can therefore be extended in the z-direction by the operators

$$\nabla_{\tau} := \frac{\partial}{\partial z} + \frac{\mu}{z} + (E\bullet)/z^2.$$

Here $E = \sum(\deg \tau^{\lambda})\tau^{\lambda}\partial/\partial \tau^{\lambda}$ is the *Euler field* and $\mu = \text{diag}(\mu_1, ..., \mu_N)$ is the *Hodge grading operator* (anti-symmetric with respect to the metric and diagonal in a graded basis of H).

The extended connection is flat (since the oscillating integrals $z\partial\mathcal{J}_{\mathfrak{B}}/\partial\tau^{\delta}$ provide a basis of flat sections) and can be considered as an *isomonodromic* family of connections ∇_{τ} over $z \in \mathbb{C}\backslash 0$ depending on the parameter $\tau \in \mathcal{T}$. In this situation, the calibration $S(z,\tau) = \mathbf{1} + S_1 z^{-1} + S_2 z^{-2} + ...$ can be chosen as a gauge transformation near $z = \infty$ of the operator ∇_{τ} to the normal form $\nabla_0 = \partial/\partial z + \mu/z$. In other words, the fundamental solution matrix $(z\partial\mathcal{J}_{\mathfrak{B}_{\lambda}}/\partial\tau^{\delta})$ consisting of oscillating integrals can be written as $S(z,\tau)\ z^{-\mu}\ C$ (where C is a constant matrix), and the J-function can be extracted from $\mathcal{J}$ in a similar way.

In singularity theory, gravitational descendents are not defined (at least mathematically) in an intrinsic way, but rather recovered from the axioms, i.e. the total descendent potential is defined as the generating function for the Lagrangian cone $\mathcal{L} \subset \mathcal{H}$ determined by the J-function.

Semisimplicity. In the case $\dim H = 1$ (interpreted either as the Gromov – Witten theory of the point target space or in mirror terms as A_1 singularity) the J-function $ze^{\tau/z}$ generates the descendent potential (in dilaton-shifted variables)

$$(10) \qquad \mathcal{F}(\mathbf{q}) = \text{ the critical value of } \frac{1}{2}\int_0^{\tau}(q_0 + q_1 x + ... + q_k\frac{x^k}{k!} + ...)^2\ dx.$$

Consider now the model Cartesian product case of a semisimple translation invariant Frobenius structure defined by the *algebra* $\mathbb{C}^N$. Let $\mathcal{L}^{(N)}$ be the corresponding Lagrangian submanifold. It is isomorphic to $(\mathcal{L}^{(1)})^N$ and is generated by direct sum $\mathcal{F}(\mathbf{q}^{(1)}) + ... + \mathcal{F}(\mathbf{q}^{(N)})$ of N copies of the function (10).

Theorem 5. *The Lagrangian submanifold $\mathcal{L} \subset \mathcal{H}$ representing the germ of a semisimple N-dimensional Frobenius manifold is locally isomorphic to $\mathcal{L}^{(N)} \subset \mathbb{C}^N((z^{-1}))$.*

Proof. This is a reformulation of a result from [24] (*cf.* Exercise 3.7 in [9] though) about existence of an asymptotical fundamental solution

$$(z\frac{\partial J^\lambda}{\partial \tau^\delta}) \sim \Psi(\tau)R(z,\tau)e^{U(\tau)/z}$$

to the system (5) at semisimple points τ. Here $R = \mathbf{1}+R_1 z+R_2 z^2+...$ is an "upper-triangular" element of the (completed) twisted loop group $\mathcal{L}^{(2)}GL_N$, $\Psi : \mathbb{C}^N \to H$ is an isomorphism of the inner product spaces, and $U = \mathrm{diag}(u_1,...,u_N)$ is the diagonal matrix of Dubrovin's *canonical coordinates* [9] on the Frobenius manifold.

In the case of K. Saito's Frobenius structures of singularity theory, $u_\lambda(\tau)$ are critical values of the Morse functions $F(\cdot,\tau)$, and an asymptotical solution $\Psi R \exp(U/z)$ can be found as the stationary phase asymptotics of the complex oscillating integrals $z\partial \mathcal{J}_{\mathfrak{B}_\lambda}/\partial \tau^\delta$.

In general, consider the cones $\mathcal{L}_\tau = S(\tau)\mathcal{L}$ representing the ancestor potentials. The operators $\exp(-U(\tau)/z)R^{-1}(\tau)\Psi^{-1}(\tau)$ act on the cones $\mathcal{L}_\tau$ despite the fact that $R = \mathbf{1} + R_1 z + ...$ are infinite z-series. This is because the ancestor potentials are tame and in particular can be considered as series in $\bar{t}_0, \bar{t}_1$ with coefficients *polynomial* in $\bar{t}_2, \bar{t}_3, ...$ (see Section 8 in [26] for more details). The resulting cone $\mathcal{L}'$ does not depend on τ since the operators S and $\Psi R \exp(U/z)$ satisfy the same differential equations (5) in τ. On the other hand, the operators R preserve the isotropic space $z\mathbb{C}^N[[z]]$ contained in (the completion of) the cones $\mathcal{L}_\tau$. This implies that the cone $\mathcal{L}'$ together with the point $-z$ contains all points $\exp(-U/z)(-z)$. Thus $z\exp(U/z)$ is the J-function of $\mathcal{L}'$ which shows that $\mathcal{L}' = \mathcal{L}^{(N)}$. $\square$

The operator $R(z,\tau)$ is unique up to the right multiplication by the automorphisms $\exp(a_0/z + a_1 z + a_2 z^3 + ...)$, $a_i \in \mathbb{C}^N$, of $\mathcal{L}^{(N)}$[5]. The ambiguity in is eliminated by imposing the additional homogeneity condition (9) (or equivalently $L_E R = 0$) available in the presence of the Euler vector field E in the definition of the Frobenius structure. This is the case in singularity theory and in (non-equivariant!) Gromov – Witten theory.

Virasoro constraints. The so called "Virasoro conjecture" was invented by T. Eguchi, K. Hori, M. Jinzenji, C.-S. Xiong and S. Katz [12, 13] and upgraded to the rank of an axiom in abstract topological field theory by B. Dubrovin and Y. Zhang [10]. In the axiomatic context, the Virasoro constraints are defined in the presence of the additional grading axiom expressing the role of the Euler field E in the definition of Frobenius structures. The genus 0 Virasoro constraints are known to contain no information in addition to the grading axiom and the system DE+SE+TRR. The geometrical argument below replaces the original proof of this fact given by X. Liu – G. Tian [38] as well as the shorter proof given by E. Getzler [18].

Suppose that linear operators A and B on a vector space are anti-symmetric with respect to a bilinear form. Then the operators $l_m = ABABA...BA$ (B repeated m times) are also anti-symmetric. On the other hand, if $AB - BA = B$, then $l_m, m = 0, 1, 2, ...$, commute as the vector fields $x^{m+1}\partial/\partial x$ on the line: $[l_m, l_n] = (n-m)l_{m+n}$.

In the symplectic space $(\mathcal{H}, \Omega)$, consider infinitesimal symplectic transformations A, B satisfying $[A, B] = B$, where $A = l_0 = zd/dz+1/2+a(z)$ with $a^*(-z)+a(z) = 0$, and B is multiplication by z.

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[5]and up to reversing or renumbering coordinates in $\mathbb{C}^N$.

The additional grading axiom for the descendent potential $\mathcal{F}$ is formulated as the invariance of the cone $\mathcal{L}$ under the flow of the linear hamiltonian vector field on $\mathcal{H}$ defined by the operator l_0 with a special choice of a (determined by the the Euler field E). For instance, in the case of weighted - homogeneous singularities $a = \mu$, and in Gromov – Witten theory take $a = \mu + \rho/z$ where μ is the Hodge grading operator on $H^*(X; \mathbb{Q})$ and ρ is the operator of multiplication by $c_1(T_X)$ in $H^*(X; \mathbb{Q})$. However the explicit form of a is irrelevant for the present discussion. Note that the vector field defined by the operator $l_{-1} = B^{-1} = z^{-1}$ (corresponding to $\partial/\partial x$) is also tangent to $\mathcal{L}$ due to the string equation. The following theorem expresses the genus 0 Virasoro constraints for $\mathcal{F}$.

Theorem 6. *Suppose the vector field on $\mathcal{H}$ defined by the operator l_0 is tangent to the Lagrangian cone $\mathcal{L}$ satisfying the condition $(\star)$. Then the same is true for the vector fields defined by the operators $l_m = l_0 z l_0 z ... z l_0$, $m = 1, 2, ...$*

Proof. Let L be the Lagrangian space tangent to $\mathcal{L}$ along zL. We know therefore that for any $\mathbf{f} \in zL$ the vector $l_0 \mathbf{f}$ is in L. This implies $z l_0 \mathbf{f} \in zL$, implies $l_0 z l_0 \mathbf{f} \in L$, implies $z l_0 z l_0 \mathbf{f} \in zL$, ..., and therefore for all $m = 1, 2, ...$ the vectors $l_m \mathbf{f}$ are tangent to the cone at the point $\mathbf{f}$. $\square$.

Higher genus and quantization. Correlators of the higher genus axiomatic theory are arranged into the Taylor coefficients of a sequence of formal functions $\mathcal{F}^{(g)}(\mathbf{t})$, $g = 0, 1, 2, ...$, called *genus g descendent potentials*, which includes the genus 0 theory with $\mathcal{F} = \mathcal{F}^{(0)}$. In this section, we collect some results from [24, 25, 7, 26, 27, 30, 31] illustrating the following general observation: *the higher genus theory is quantization of the genus 0 theory and inherits the twisted loop group of symmetries.*

The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = T^*\mathcal{H}_+$ allows one to quantize quadratic hamiltonians on the symplectic space $(\mathcal{H}, \Omega)$ using the following standard rules:

$$(q_\alpha q_\beta)\hat{} = q_\alpha q_\beta/\hbar, \; (q_\alpha p_\beta)\hat{} = q_\alpha \partial/\partial q_\beta, \; (p_\alpha p_\beta)\hat{} = \hbar \partial^2/\partial q_\alpha \partial q_\beta.$$

Quantized quadratic hamiltonians act on functions of $\mathbf{q} \in \mathcal{H}_+$ depending on the parameter $\hbar$. They form a *projective* representation of the Lie algebra of infinitesimal symplectic transformations and of its Lie subalgebra $\mathcal{L}^{(2)}gl(H)$. The notation $\hat{A}$ for a simplectic transformation $A \in \mathcal{L}^{(2)}GL(H)$ will be understood as the operator $\exp(\ln A)\hat{}$.

The functions $\mathcal{F}^{(g)}$ are assembled into the *total descendent potential*

$$\mathcal{D} = e^{\mathcal{F}^{(0)}/\hbar + \mathcal{F}^{(1)} + \hbar\mathcal{F}^{(2)} + \hbar^2\mathcal{F}^{(3)} + ...}$$

Expressions of this form where $\mathcal{F}^{(g)}$ are formal functions of $\mathbf{t}$ will be called *asymptotical functions* and, after the dilaton shift $\mathbf{q}(z) = -z + t_0 + t_1 z + ...$, considered as "asymptotical elements of the Fock space". This means that we treat the 1-dimensional subspace $\langle \mathcal{D} \rangle$ spanned by $\mathcal{D}$ as a point in the projective space — the potential domain of our quantization representation.

• Theorem 3 above is derived in [7] as the quasi-classical limit $\hbar \to 0$ of the higher genus "Quantum Riemann – Roch Theorem" expressing twisted Gromov – Witten invariants via the untwisted ones:

$$\langle \mathcal{D}^{tw} \rangle = \hat{\triangle} \langle \mathcal{D} \rangle$$

where $\triangle$ is the symplectic transformation in Theorem 3. This result is based on Mumford's Grothendieck – Riemann – Roch formula [41] applied to the universal

families of stable maps and generalizes the result of Faber – Pandharipande [15] for Hodge integrals corresponding to the case of the trivial twisting bundle E.

• The higher genus formula of Kontsevich – Manin [35] (see also [19]) relating descendents and ancestors in Gromov – Witten theory has been rewritten in [25] (see also [7]) as the quantization of the calibration operators $S(\tau)$ of Theorem 2:

$$\mathcal{A}_\tau = \hat{S}(\tau)\mathcal{D},$$

where $\mathcal{A}_\tau = \exp \sum \hbar^{g-1} \bar{\mathcal{F}}_\tau^{(g)}(\bar{\mathbf{t}})$ is the total ancestor potential.

• According to [24, 25], the machinery of fixed point localization and summation over graphs used by M. Kontsevich [34] for computation of equivariant Gromov – Witten invariants in the case of tori acting on the target spaces with isolated fixed points gives rise to the following formula for the equivariant total ancestor potentials:

$$(11) \qquad \langle \mathcal{A}_\tau \rangle = \Psi(\tau)\ \hat{R}(\tau)\ \exp(U/z)\hat{\ }\ \langle \mathcal{D}_{pt}^{\otimes N} \rangle.$$

Here the rightmost function is the product $\mathcal{D}_{pt}(\mathbf{q}^{(1)})...\mathcal{D}_{pt}^{(N)}$ of $N = \dim H$ copies of the total descendent potential $\mathcal{D}_{pt}$ of the one-point space $X = pt$, and $\Psi R \exp(U/z)$ is the asymptotical solution to the system (5).[6]

• The formula (11) makes sense for any *semisimple* Frobenius structure and thus can be used as a definition of the higher genus ancestor potentials in the abstract theory. According to [25] this definition meets a number of expectations. Namely:

  ∘ The total ancestor potential $\mathcal{A}_\tau$ thus defined is *tame* in the sense that the corresponding functions $\bar{\mathcal{F}}_\tau^{(g)}(\bar{\mathbf{t}})$ satisfy

$$\frac{\partial}{\partial \bar{t}_{k_1}^{\alpha_1}} \cdots \frac{\partial}{\partial \bar{t}_{k_r}^{\alpha_r}}\ \bar{\mathcal{F}}_\tau^{(g)}\ |_{\bar{\mathbf{t}}=0} = 0 \ \ \text{whenever} \ k_1 + ... + k_r > r + 3g - 3.$$

  ∘ The corresponding total descendent potential *defined* by $\langle \mathcal{D} \rangle = \hat{S}^{-1}(\tau)\langle \mathcal{A}_\tau \rangle$ does not depend on τ. (Together with the previous property this means that $\mathcal{D}$ satisfies the $3g - 2$-jet condition of Eguchi – Xiong [14, 19] playing the role of TRR in the higher genus theory.)

  ∘ The string equation $\hat{l}_{-1}\mathcal{D} = 0$ always holds true, and the higher Virasoro constraints $\hat{l}_m\mathcal{D} = 0, m = 1, 2, ...$ hold true if the grading condition $\hat{l}_0\mathcal{D} \in \langle \mathcal{D} \rangle$ is satisfied.

• In non-equivariant Gromov–Witten theory, the formula

$$(12) \qquad \langle \mathcal{D} \rangle = \hat{S}^{-1}\ \Psi\ \hat{R}\ e^{(U/z)\hat{\ }}\ \langle \mathcal{D}_{pt}^{\otimes N} \rangle$$

is known to be correct for complex projective spaces and other Fano toric manifolds [25] and for complete flag manifolds [31] (which in particular implies that the Virasoro conjecture holds true for these manifolds). Also, it is an easy exercise on application of the dilaton equation for $\mathcal{D}_{pt}$ to check that the result of Jarvis – Kimura [30] computing gravitational descendents in the Gromov – Witten theory on the quotient *orbifold* of the one - point space by a finite group agrees with (12). In this example, the Frobenius structure is translation invariant, $(H, \circ, (\cdot, \cdot))$ is the center of the group ring, Ψ describes its orthonormal diagonalization while $S = R = \mathbf{1}$.

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[6]See [24, 25] for the explanation of how to fix the ambiguity in the construction of the asymptotical solution in the equivariant quantum cohomology theory lacking the Euler vector field E.

• According to the part of Witten's conjectures [44] proved by M. Kontsevich [33] (see also [42]) the asymptotical function $\mathcal{D}_{pt}$ describing intersection theory on the Deligne – Mumford spaces $\overline{\mathcal{M}}_{g,m}$ satisfies the KdV hierarchy of evolution equations with the time variables $t_0, t_1, t_2, ...$. The remaining part of the conjectures similarly identifies the total descendent potential in the moduli theory of curves equipped with *n-spin structures* with solutions to the nKdV (or *Gelfand – Dickey*) hierarchies of integrable systems. Corresponding Frobenius structure coincides [44] with K. Saito's structure on the miniversal deformation of the A_{n-1}-singularity. It is a result of [26] that in the case of A_{n-1}-singularity the function (12) indeed satisfies the equations of the nKdV hierarchy. In [27] this result is generalized to the *ADE*-singularities.

Conventional wisdom says that the structure of the axiomatic higher genus theory should reflect the geometry of the Deligne – Mumford spaces, and vice versa. Yet both subjects are far from being clear. In view of the above examples it is tempting to impose the following two requirements on conjectural axioms of the higher genus theory.

A. The set of asymptotical elements of the Fock space satisfying the axioms has to be invariant under the quantization representation of the twisted loop Lie algebra.

In particular, the stabilizer of a Lagrangian cone $\mathcal{L}$ with the generating function $\mathcal{F}$ satisfying the axioms of the genus 0 theory has to act on the set of asymptotical elements satisfying the axioms of the higher genus theory and having the same genus 0 part $\mathcal{F}^{(0)} = \mathcal{F}$.

B. The action of the stabilizer is transitive.

The latter requirement is inspired by the uniqueness lemma of Dubrovin – Zhang [11] which says that in the *semisimple* case an asymptotical function $\mathcal{D}$ satisfying the higher genus TRR and the Virasoro constraints is unique (and thus coincides with (12) according to the results quoted earlier). It would be preferable however to derive the Virasoro constraints from other axioms (such as or similar to A+B) just the way it is done in the proof of Theorem 6 in the genus 0 theory.

Quantum K-theory. The material of this section represents our joint work with Tom Coates and shows that the description of the correlators and their properties in geometrical terms of Lagrangian cones and quantization remains valid in the context of quantum K-theory.

In quantum K-theory, a basis of "observables" has the form $\Phi_\alpha L_i^k$ where $\{\Phi^\alpha\}$ is a linear $\mathbb{Q}$-basis in $K^*(X)$ (here $K^0(X)$ is the Grothendieck group of complex vector bundles over the target space X) and $L_1, ..., L_n$ are the universal cotangent line bundles over the moduli spaces of stable maps $X_{g,n,d}$.

Correlators can be defined in cohomological terms using the *virtual tangent bundles* $T_{g,n,d}$ of the moduli spaces $X_{g,n,d}$ and with holomorphic Euler characteristics in mind:

$$(13) \quad \langle \Phi^{\alpha_1} L^{k_1}, ..., \Phi^{\alpha_n} L^{k_n} \rangle_{g,n,d} = \int_{[X_{g,n,d}]} \text{td}(T_{g,n,d}) \ \text{ch}_*[\otimes_{i=1}^n \text{ev}_i^*(\Phi^{\alpha_i}) \otimes L_i^{\otimes k_i}].$$

First properties of K-theoretic Gromov – Witten invariants are discussed in [23], and the foundations in the setting of algebraic target spaces have been laid down by Yuan-Pin Lee in [37]. We have to stress that the correlators (13) are only approximations to the actual holomorphic Euler characteristics which in the orbifold / orbibundle context are given by Kawasaki's Hirzebruch – Riemann – Roch Theorem

[32]. The correlators (13) differ therefore from those used in the papers [23, 37]. However the general *properties* of the correlators (as opposed to their *values*) remain the same as in [23, 37]. This is because the bundles, sheaves and their properties used in those papers are the same as in the "fake" version of K-theory considered here.

In complete analogy with the cohomology theory, one can introduce K-theoretic descendent potentials $\mathcal{F}^{(g)}(\mathbf{t})$ as formal functions of $t_0 + t_1 L + t_2 L^2 + ...$ with $t_i \in K := K^*(X) \otimes \mathbb{Q}[[Q]]$. As it was found in [23], K-theoretic genus 0 Gromov – Witten invariants define a "Frobenius-like" structure on K which however exhibits the following peculiar distinction from the case of cohomology theory. The constant coefficient metric

$$g_{\alpha\beta} = \langle \Phi_\alpha, \mathbf{1}, \Phi_\beta \rangle_{0,3,0} = \chi(X; \Phi_\alpha \otimes \Phi_\beta) = \int_X \mathrm{td}(T_X) \ \mathrm{ch}_*(\Phi_\alpha) \ \mathrm{ch}_*(\Phi_\beta)$$

has to be replaced in all formulas by the non-constant one:

$$G_{\alpha\beta}(\tau) = g_{\alpha\beta} + \sum_{n \geq 0, d} \frac{Q^d}{n!} \langle \Phi_\alpha, \tau, ..., \tau, \Phi_\beta \rangle_{0,n,d}, \quad \text{where } \tau = \sum \tau^\mu \Phi_\mu \in K.$$

The multiplication $\Phi_\delta \bullet \Phi_\lambda = \sum_\mu A^\mu_{\delta\lambda} \Phi_\mu$, where

$$A^\mu_{\delta\lambda}(\tau) = \sum_\nu G^{\mu\nu}(\tau) \ \partial_{\delta,0} \partial_{\lambda,0} \partial_{\nu,0} \mathcal{F}^{(0)}|_{\mathbf{t}=(\tau,0,...)}, \quad \text{and } (G^{\mu\nu}) = (G_{\mu\nu})^{-1},$$

makes the tangent spaces $T_\tau K$ Frobenius algebras with respect to the inner product $(G_{\mu\nu})$. The K-theory version of the J-function (7)

$$(14) \qquad J = (1-q) + \tau + \sum_{n \geq 0, d} \frac{Q^d}{n!} (\mathrm{ev}_{n+1})^K_* \left[\frac{\otimes^n_{i=1} \mathrm{ev}^*_i(\tau)}{\mathbf{1} - q L_{n+1}} \right]$$

satisfies the system

$$(15) \qquad (1-q) \frac{\partial}{\partial \tau^\delta} \frac{\partial}{\partial \tau^\lambda} J = \sum A^\mu_{\delta\lambda} \frac{\partial}{\partial \tau^\mu} \ J.$$

The system (15) at $q = -1$ yields the Levi – Civita connection of the metric $(G_{\mu\nu})$ which is therefore flat.

We will show that it is possible to adapt our language of symplectic loop spaces to absorb these changes at least in the version of quantum K-theory discussed here.[7]

First, note that the J-function (14) considered as a $1/(1-q)$-series becomes finite when reduced modulo Q^d since $L_{n+1} - 1$ is nilpotent in $K^0(X_{0,n,d})$. We will call such Laurent $1/(1-q)$-series *convergent* away from $q = 1$ in the Q-adic topology (i.e. for those $q \in \mathbb{Q}[[Q]]$ whose Q-adic distance to 1 is ≥ 1).

Next, consider the loop space $\mathcal{K}$ of all vector Laurent series with coefficients in K convergent in this sense away from $q = 1$. Equip K with the inner product $(\Phi_\alpha, \Phi_\beta) = g_{\alpha\beta}$ (the original constant one!) and define in $\mathcal{K}$ the symplectic form

$$\Omega(\mathbf{f}, \mathbf{g}) = \frac{1}{2\pi i} \oint (\mathbf{f}(q^{-1}), \mathbf{g}(q)) \ \frac{dq}{q} := -[\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}](\mathbf{f}(q^{-1}), \mathbf{g}(q)) \ \frac{dq}{q}.$$

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[7]In the original version of quantum K-theory studied in [23, 37] the orbifold features of the moduli spaces $X_{g,n,d}$ play prominent role. For example, the classes $L_i \in K^0(X_{g,n,d})$ no longer behave unipotently. This causes some difficulties which seem formal at first (so that an analogue of Theorem 7 below still holds true), but become overwhelming in the more sophisticated Riemann – Roch problems discussed in the next section.

Substituting $q = \exp z$ (as motivated by $\mathrm{ch}_*(L_{n+1}) = \exp \psi_{n+1}$) we find that this Ω is in a sense the same as in cohomology theory. Consider the polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ where $\mathcal{K}_+ = K[q]$ and $\mathcal{K}_-$ consists of the series convergent away from $q = 1$ *and vanishing at* $q = \infty$. The polarization is Lagrangian (although not invariant with respect to $z \mapsto -z$) and identifies $\mathcal{K}$ with (a topologized version of) $T^*\mathcal{K}_+$. We use the dilaton shift $\mathbf{q}(q) = 1 - q + t_0 + t_1 q + t_2 q^2 + ...$ to encode the genus 0 descendent potential by a Lagrangian submanifolds $\mathcal{L}$, and furthermore — to identify the total descendent and ancestor potentials of higher genus quantum K-theory with the appropriate asymptotical functions $\mathcal{D}$ near $\mathbf{q} = (1 - q) \in \mathcal{K}_+$.

Now, repeat the construction for ancestors $\mathcal{L}_\tau$, $\mathcal{A}_\tau$ using the inner product $(\Phi_\alpha, \Phi_\beta)_\tau = G_{\alpha\beta}(\tau)$ and the corresponding symplectic structure Ω^τ in the space $\mathcal{K}^\tau = \mathcal{K}$.

It turns out that in quantum K-theory the calibration operators $S(q, \tau)$ defined via the J-function (14) as

$$S_\beta^\alpha(q, \tau) = \sum_{\mu\nu} G^{\alpha\mu}(\tau)\, \frac{\partial J^\nu}{\partial \tau^\mu}(q, \tau)\, g_{\nu\beta}$$

satisfy

$$\sum_{\alpha\beta} S_\mu^\alpha(q, \tau) G^{\alpha\beta}(\tau) S_\nu^\beta(q^{-1}, \tau) = g_{\mu\nu}.$$

This shows that $S(\tau)$ is symplectic as a linear map from $(\mathcal{K}, \Omega)$ to $(\mathcal{K}^\tau, \Omega^\tau)$. The following result is the K-theoretic version of the formula [25, 7] relating descendents and ancestors.

Theorem 7. *We have* $e^{F^{(1)}(\tau)}\mathcal{A}_\tau = \hat{S}(\tau)\mathcal{D}$ *and in particular* $\mathcal{L}_\tau = S(\tau)\mathcal{L}$.

Since $\mathrm{ch}_*(\bar{L}_i - 1) = e^{\bar{\psi}_i} - 1$ are nilpotent in $H^*(\overline{\mathcal{M}}_{0,n}; \mathbb{Q})$, the Lagrangian sections $\mathcal{L}_\tau$ are tangent to $\mathcal{K}_+^\tau$ along $(1 - q)\mathcal{K}_+^\tau$.

Corollary. *The Lagrangian submanifold* $\mathcal{L} \subset \mathcal{K}$ *is a cone with the vertex at the origin and satisfies the condition that its tangent spaces L are tangent to $\mathcal{L}$ exactly along* $(1 - q)L$.

The conic property of $\mathcal{L}$ is equivalent to the genus 0 case of the K-theoretic *dilaton equation*

$$\langle \mathbf{t}^{(1)}(L), ..., \mathbf{t}^{(n)}(L), 1 - L \rangle_{g,n+1,d} = (2 - 2g - n)\langle \mathbf{t}^{(1)}(L), ..., \mathbf{t}^{(n)}(L) \rangle_{g,n,d},$$

where $\mathbf{t}^{(i)}$ are arbitrary polynomials of the universal cotangent line bundles $L = L_i$.

The operator of multiplication by

$$\frac{1}{1 - q} - \frac{1}{2} = \frac{1}{2}\,\frac{1 + q}{1 - q}$$

is anti-symmetric with respect to Ω and defines a linear hamiltonian vector field on $(\mathcal{K}, \Omega)$ which is tangent to the cone $\mathcal{L}$. This property of $\mathcal{L}$ expresses the following genus 0 K-theoretic *string equation*

$$\langle \mathbf{t}^{(1)}(L), ..., \mathbf{t}^{(n)}(L), 1 \rangle_{0,n+1,d} =$$

$$\langle \mathbf{t}^{(1)}(L), ..., \mathbf{t}^{(n)}(L) \rangle_{0,n,d} + \sum_{i=1}^n \langle \mathbf{t}^{(1)}(L), ..., \frac{\mathbf{t}^{(i)}(L) - \mathbf{t}^{(i)}(1)}{L - 1}, ..., \mathbf{t}^{(n)}(L) \rangle_{0,n,d}.$$

As we mentioned earlier, the same results hold true in the original version of the quantum K-theory studied in [23, 37].

Quantum cobordism theory. The complex cobordism theory $MU^*(\cdot)$ is defined in terms of homotopy classes of maps to the spectrum $MU(k)$ of the Thom spaces of universal $U_{k/2}$-bundles:

$$MU^n(B) = \lim_{k \mapsto \infty} \pi(\Sigma^k B, MU(n+k)).$$

When B is a stably almost complex manifold of real dimension m, the famous Pontryagin – Thom construction identifies elements of $MU^n(B)$ with appropriately framed bordism classes of maps $M \to B$ of stably almost complex manifolds M of real dimension $m-n$. This identification plays the role of the Poincaré isomorphism. Similarly to the complex K-theory, there is the *Chern – Dold character* which provides natural multiplicative isomorphisms

$$\mathrm{Ch} : MU^*(M) \otimes \mathbb{Q} \to H^*(M, \Lambda^*).$$

Here $\Lambda^* = MU^*(pt) \otimes \mathbb{Q}$ is the coefficient ring of the theory and is isomorphic to the polynomial algebra on the generators of degrees $-2k$ Poincaré – dual to the bordism calsses $[\mathbb{C}P^k]$. The theory MU^* is known to be the universal extraordinary cohomology theory where complex vector bundles are oriented. Orientation of complex bundles in MU^* is uniquely determined by the cobordism-valued Euler class $u \in MU^2(\mathbb{C}P^\infty)$ of the universal complex line bundle. Explicitly, the Euler class of $\mathcal{O}(1)$ over $\mathbb{C}P^N$ is Poincaré – dual to the embedding $\mathbb{C}P^{N-1} \to \mathbb{C}P^N$ of a hyperplane section. The image of u under the Chern – Dold character has the form

(16) $$u(z) = z + a_1 z^2 + a_2 z^3 + ...$$

where z is the cohomological 1-st Chern of the universal line bundle $\mathcal{O}(1)$, and $\{a_k\}$ is another set of generators in Λ^*. The operation of tensor product of line bundles with the Euler classes v and w defines a formal group law $F(v, w)$ on $MU^*(\mathbb{C}P^\infty) = \Lambda^*[[u]]$. The series $u(z)$ is interpreted as an isomorphism with the additive group $(x, y) \mapsto x + y$: $F(v, w) = u(z(v) + z(w))$. Here $z(\cdot)$ is the series inverse to $u(z)$. It is known as the *logarithm* of the formal group and explicitly takes on the form

$$z = u + [\mathbb{C}P^1]\frac{u^2}{2} + [\mathbb{C}P^2]\frac{u^3}{3} + [\mathbb{C}P^3]\frac{u^4}{4} + ...$$

Specialization of the parameters $[\mathbb{C}P^k] \mapsto 0$ yields the cohomology theory, and $[\mathbb{C}P^k] \mapsto 1$ yields the complex K-theory. In the latter example, $z = -\ln(1 - u)$ and hence $u(z) = 1 - \exp(-z)$. Similarly to the K-theory, one can compute push-forwards in MU^*-theory in terms of cohomology theory. In particular, for a stably almost complex manifold B, we have the *Hirzebruch – Riemann – Roch formula*

(17) $$\forall c \in MU^*(B), \quad \pi_*^{MU}(c) = \int_B \mathrm{Ch}(c) \; \mathrm{Td}(T_B) \; \in \; \Lambda^*,$$

where $\pi : B \to pt$, and $\mathrm{Td}(T_B)$ is the *Todd genus* of the tangent bundle. It is characterized as the only multiplicative characteristic class which for the universal line bundle is equal to

$$\mathrm{Td} = \frac{z}{u(z)} = \exp \sum_{k=1}^\infty s_k \frac{z^k}{k!}.$$

Here $s_1, s_2, ...$ is one more set of generators in Λ^*. To round up the introduction, let us mention the Landweber – Novikov algebra of stable cohomological operations in

complex cobordism theory. The operations correspond to cobordism-valued characteristic classes σ of complex vector bundles. To apply such an operation to the cobordism element of B Poincaré-dual to a map $\pi : M \to B$ of stably almost complex manifolds, one takes the relative normal bundle $\pi^* T_B \ominus T_M$ over M and pushes-forward its characteristic class σ to B. According to Buchshtaber – Shokurov [5], after tensoring with $\mathbb{Q}$ the Landweber – Novikov algebra becomes isomorphic to the algebra of left-invariant differential operators on the group of diffeomorphisms (16). The Landweber – Novikov operations commute with the Chern – Dold character and can therefore be expressed in the cohomology theory $H^*(\cdot, \Lambda^*)$ as certain differential operators on the algebra Λ^* of functions on the group. The algebra is generated by the derivations L_n whose action on the generators a_k is given by

$$(L_n u)(z) = u(z)^{n+1}, \ \ \text{or, equivalently,} \ \ L_n = u^{n+1} \partial/\partial u, \ n = 1, 2, ...$$

These generators correspond to the characteristic classes defined by the Newton polynomials

$$u_1^n + u_2^n + ... \in MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times ...).$$

In a sense, the idea of Gromov – Witten invariants with values in cobordisms is already present in Gromov's original philosophy [28] of symplectic invariants to be constructed as bordism invariants of spaces of pseudo-holomorphic curves. The possibility to define Gromov – Witten invariants with values in the cobordism ring Λ^* is mentioned in Kontsevich's work [34]. This proposal was further advanced by Morava [40] in a hope to explain the Virasoro constraints in terms of the Landweber – Novikov operations. Following suggestions of these authors, we define cobordism-valued Gromov – Witten invariants via (17) using the virtual tangent bundles $T_{g,n,d}$ of the moduli spaces $X_{g,n,d}$. We formulate below the genus 0 version of the "Quantum Hirzebruch – Riemann – Roch Theorem" which expresses these new Gromov – Witten invariants via the old ones. The discussion below represents joint work of Tom Coates and the author. The forthcoming thesis [6] contains many details omitted here.

In quantum cobordism theory, a basis of observables has the form $\Phi_\alpha u_i^k$ where $\{\Phi^\alpha\}$ is a basis of $H\Lambda := MU^*(X) \otimes \mathbb{Q}[[Q]]$ over $\Lambda^* \otimes \mathbb{Q}[[Q]]$, and u_i is the Euler class of the universal cotangent line bundle L_i. The correlators are defined by the formula
(18)

$$\langle \Phi_{\alpha_1} u^{k_1}, ..., \Phi_{\alpha_n} u^{k_n} \rangle_{g,n,d} := \int_{[X_{g,n,d}]} e^{\sum_{k=1}^\infty s_k \operatorname{ch}_k(T_{g,n,d})} \prod_{i=1}^n \left[\operatorname{ev}_i^* \operatorname{Ch}(\Phi_{\alpha_i}) \ u(\psi_i)^{k_i} \right].$$

One can introduce Λ^*-valued descendent potentials $\mathcal{F}^{(g)}(\mathbf{t})$ in complete analogy with the cohomology theory as formal functions of $t_0 + t_1 u + t_2 u^2 + ...$ with $t_i \in H\Lambda$.

In order to express the cobordism-valued Gromov – Witten invariants in terms of cohomological ones one needs to control the classes $\operatorname{ch}_k(T_{g,n,d})$. Roughly speaking, the virtual tangent bundles $T_{g,n,d}$ consist of two parts. One of them, representing deformations of holomorphic *maps* of Riemann surfaces with a fixed complex structure, coincides with $E_{g,n,d}$ (as in Theorem 3 about twisted Gromov – Witten invariants) with $E = T_X$. The effect of this part can be described therefore via Theorem 3. The other part represents deformations of complex structures and contributes to the correlators in a complicated fashion. Surprisingly, the very formalism of the symplectic loop space takes effective care of these contributions.

We denote by $\Lambda = \mathbb{Q}\{s_1, s_2, ...\}$ the coefficient ring of complex cobordism theory completed with respect to the **s**-adic norm defined by the grading $\deg s_k = -k$. Let $H\Lambda$ be the cobordism group of the target space X over Λ equipped with the inner product

$$(a, b)_\mathbf{s} := \pi_*^{MU}(ab) = \int_X \mathrm{Ch}(a)\, \mathrm{Ch}(b) \exp\left(\sum s_k \, \mathrm{ch}_k(T_X)\right).$$

Denote $\mathcal{H}\Lambda$ the loop space $H\Lambda\{\{u^{-1}\}\}$ of Laurent series $\sum f_k u^k$ with coefficients $f_k \in H\Lambda$ which can be non-zero for all $k \in \mathbb{Z}$ but should satisfy the condition that $\mathbf{f}_k \to 0$ in the **s**-adic topology as $k \to +\infty$. Introduce the symplectic form with values in Λ:

$$\Omega_\mathbf{s}(\mathbf{f}, \mathbf{g}) := \frac{1}{2\pi i} \oint \pi_*^{MU}(\mathbf{f}(u^*)\, \mathbf{g}(u)) \sum_{k \geq 0} [\mathbb{C}P^k] u^k \, du = \frac{1}{2\pi i} \oint (\mathbf{f}(u(-z)), \mathbf{g}(u(z))_\mathbf{s} \, dz.$$

The following *quantum Chern – Dold character* identifies the symplectic structure $\Omega_\mathbf{s}$ with its cohomological version $\Omega = \Omega_\mathbf{0}$:

$$\mathrm{qCh} : \mathcal{H}\Lambda \to \mathcal{H} = \lambda \otimes H((z^{-1})) : \quad \mathrm{qCh}(\mathbf{f}) = \sqrt{\mathrm{Td}(T_X)} \sum_{k \in \mathbb{Z}} \mathrm{Ch}(f_k) u^k(z).$$

Assuming $|\psi| < |z|$ write

$$\frac{1}{u(-z - \psi)} = \sum_{k \geq 0} u(\psi)^k v_k(u(z)) = \sum_{k \geq 0} \psi^k w_k(z).$$

Put

$$\mathcal{H}\Lambda_+ = H\Lambda\{u\} = \left\{ \text{ power series } \sum q_k u^k \text{ with } q_k \to 0 \right\}$$

and

$$\mathcal{H}\Lambda_- = \left\{ \text{ arbitrary infinite series } \sum p_k v_k(u) \right\}.$$

The following residue computation shows that $\{..., u^k, ..., v_k(u), ...\}$ (as well as $\{..., z^k, ..., w_k(z), ...\}$) is a Darboux basis in $(\mathcal{H}\Lambda, \Omega_\mathbf{s})$ and implies that the spaces $\mathcal{H}\Lambda_\pm$ form a Lagrangian polarization (depending on **s**):

$$\frac{1}{2\pi i} \oint \frac{dz}{u(z - x)\, u(-z - y)} = \begin{cases} 0 & \text{if} \quad |z| < |x|, |y| \text{ or } |x|, |y| < |z| \\ \pm\frac{1}{u(-x-y)} & \text{if} \quad |x| < |z| < |y| \text{ or } |y| < |z| < |x| \end{cases}.$$

Introduce the dilaton shift convention

$$\mathbf{q}(u) = \mathbf{t}(u) + u^*(u), \quad \text{where} \quad u^*(u(z)) = u(-z).$$

We consider the genus 0 descendent potential $\mathcal{F}^{(0)}$ of the quantum cobordism theory as a function depending formally on the parameters $\mathbf{s} = (s_1, s_2, ...)$ and define the corresponding family of Lagrangian submanifolds $\mathcal{L}_\mathbf{s}$ in the family of symplectic spaces $T^*\mathcal{H}\Lambda_+ = \mathcal{H}\Lambda$ over $\mathrm{Spec}\mathcal{L}$.

The following theorem expresses $\mathcal{L}_\mathbf{s}$ in terms of the cone $\mathcal{L} = \mathcal{L}_\mathbf{0}$ describing the quantum cohomology theory of X.

Theorem 8 (see [6]). *The image of $\mathcal{L}_\mathbf{s}$ under the quantum Chern – Dold character coincides with the Lagrangian cone of the Gromov – Witten theory on X twisted by the class $\mathrm{Td}(T_X)$:*

$$\mathrm{qCh}(\mathcal{L}_\mathbf{s}) = \square\mathcal{L}, \quad \square \sim \sqrt{\mathrm{Td}(T_X)} \prod_{m=1}^\infty \mathrm{Td}(T_X \otimes L^{-m}),$$

where L is a line bundle with $c_1(L) = z$.

In fact $\mathrm{qCh}(\mathcal{H}\Lambda_+) = \mathcal{H}_+$, but $\mathrm{qCh}(\mathcal{H}\Lambda_-)$ and the dilaton shift depend on $\mathbf{s}$. This causes a discrepancy between the descendent potentials of the quantum cobordism theory and the twisted theory which accounts for the entire contribution of variations of complex structures on Riemann surfaces into the virtual tangent bundles of the moduli spaces of stable maps.

The same happens in the higher genus theory. The quantum Chern – Dold character identifies the Heisenberg Lie algebras of the spaces $(\mathcal{H}\Lambda, \Omega_{\mathbf{s}})$. By the virtue of the Stone – von Neumann theorem and the Schur lemma, the Fock spaces corresponding to different polarizations are identified projectively. The total descendent potentials $\mathcal{D}_{\mathbf{s}}$ of the quantum cobordism theory differ from the total descendent potentials $\mathcal{D}_{\mathbf{s}}^{tw}$ of the appropriately twisted Gromov – Witten theory in such a way that the corresponding lines $\langle \mathcal{D}_{\mathbf{s}} \rangle$ and $\langle \mathcal{D}_{\mathbf{s}}^{tw} \rangle$ in the representation space of the Heisenberg algebra coincide.

Returning to the genus 0 case, notice that in the case of translation invariant Frobenius structure the cone $\mathcal{L}$ is invariant under the symplectic transformations defined by $\square$.

Corollary. *When $X = pt$, we have $\mathrm{qCh}(\mathcal{L}_{\mathbf{s}}) = \mathcal{L}$.*

The formal group law

$$(19) \qquad u(x + y) = u(x) + u(y) - \lambda u(x)u(y), \quad u(z) = (1 - e^{-\lambda z})/\lambda,$$

interpolates between cohomology and K-theory. Thus the specialization $[\mathbb{C}P^k] \mapsto \lambda^k$ allows one to adjust the quantum Hisrzebruch – Riemann – Roch theorem to the case of quantum K-theory. (The correlators discussed in the previous section correspond to $\lambda = -1$.)

Corollary. *The quantum Chern character* $\mathrm{qch} : \mathcal{K} \mapsto \mathcal{H}$ *defined via* $\mathrm{qch}\, u = (1 - \exp(-\lambda z))/\lambda$ *identifies* $\mathrm{qch}(\mathcal{L}_\lambda)$ *with* $\square \mathcal{L}$ *where* $\square$ *is defined via the Todd class* $\mathrm{td}(L) = \lambda z/(1 - \exp(-\lambda z))$.

The quantum Chern – Dold character transforms multiplication by u to multiplication by $u(z) = z + ... \in \Lambda\{z\}$. Transformations defined by $\square$ belong to the twisted loop group and thus commute with multiplication by $u(z)$.

Corollary. *The submanifolds* $\mathcal{L}_{\mathbf{s}} \subset \mathcal{H}\Lambda$ *are Lagrangian cones satisfying the condition* $(\star)$.

In particular, the dilaton equation holds true in quantum cobordism theory (which is easy to prove directly for any genus). Moreover, the cubical form on L/zL defined by the correlators $\langle \Phi_\alpha, \Phi_\beta, \Phi_\gamma \rangle$ represents the *Yukawa coupling* defined on any Lagrangian submanifold in a linear symplectic space (see [20]). The form coincides therefore with the structure tensor of the Frobenius manifold defined by the twisted Gromov – Witten theory. However in quantum cobordism theory, there is no a simple formula for the string equation, as there is no general reasons for flatness of the metric $G_{\alpha\beta} = \langle \Phi_\alpha, 1, \Phi_\beta \rangle$ or associativity of the quantum cup-product whose constructions depend on the polarization. In fact, a key step — Barannikov's derivation of the equation (5) for the J-function — was based on the relation $u^* \mathcal{H}\Lambda_- = H\Lambda + \mathcal{H}\Lambda_-$. It is not hard to see that the group laws (19) are the only ones satisfying this condition.

Indeed, the inclusion $\forall k \geq 0, \ Hu^*w_k \subset H\Lambda + \mathcal{H}\Lambda_-$ means that the projection of all $u(-z)w_k(z)$ along the subspace $Span(w_l(z), l = 0, 1, 2, ...)$ yields constant

polynomials. By definition we have $\sum_l w_l(-z)x^l = 1/u(z-x)$ when $|x| < |z|$ and $\sum_k u(-z)w_k(z)y^k = u(-z)/u(-z-y)$ when $|y| < |z|$. In terms of these generating functions the inclusion is equivalent therefore to the condition that

$$\frac{1}{2\pi i} \oint_{|x|,|y|<|z|} \frac{u(-z)\ dz}{u(z-x)\ u(-z-y)} = \frac{u(-x)}{u(-x-y)} - \frac{u(y)}{u(-x-y)}$$

is independent of x for all y. Differenting $u(-x-y)f(y) = u(-x) - u(y)$ in x we find $u'(-x-y)f(y) = u'(-x)$ which at $x = 0$ yields $f(y) = 1/u'(-y)$ (since $u'(0) = 1$) and implies $u'(-x-y) = u'(-x)u'(-y)$. Thus $u'(-x) = \exp(\lambda x)$ and respectively $u(z) = (1 - \exp(-\lambda z))/\lambda$.

It would be interesting to find out if ellipticity of $u(z)$ brings any "improvements" in properties of *elliptic quantum cohomology* in comparison with the general quantum cobordism theory.

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