Elliptic Gromov-Witten invariants
and the generalized mirror conjecture.

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Abstract

A conjecture expressing genus 1 Gromov-Witten invariants in mirror-theoretic terms of semi-simple Frobenius structures and complex oscillating integrals is formulated. The proof of the conjecture is given for torus-equivariant Gromov - Witten invariants of compact Kähler manifolds with isolated fixed points and for concave bundle spaces over such manifolds. Several results on genus 0 Gromov - Witten theory include: a non-linear Serre duality theorem, its application to the genus 0 mirror conjecture, a mirror theorem for concave bundle spaces over toric manifolds generalizing a recent result of B. Lian, K. Liu and S.-T. Yau. We also establish a correspondence (see the extensive footnote in section 4) between their new proof of the genus 0 mirror conjecture for quintic 3-folds and our proof of the same conjecture given two years ago.

Introduction

Gromov-Witten invariants of a compact symplectic manifold X are defined by means of enumeration of compact pseudo-holomorphic curves in X. For any cycle M in the moduli space of genus g Riemann surfaces with n marked points and for any n cycles in X one can define a GW-invariant counting those genus g marked pseudo-holomorphic curves in X which pass by the marked points through the given cycles in X and whose holomorphic type belongs to M. The handful of GW-invariants thus introduced obeys various universal identities which originate from topology of moduli spaces of Riemann surfaces and constitute a remarkable and fairly sophisticated algebraic structure. In this paper, we study the structure formed by rational and elliptic GW-invariants.

The structure of rational GW-invariants alone is well-understood and has been formalized by B. Dubrovin [7] in the concept of Frobenius manifolds. The genus 0 GW-invariants define on the total cohomology space $H := H^*(X)$ a Frobenius manifold structure; roughly speaking, it consists of the associative commutative [1] *quantum cup-product* on the tangent spaces $T_t H$ which is a deformation of the ordinary cup-product, is symmetric with respect to the Poincare intersection form $\langle \cdot, \cdot \rangle$ and depends on the application point $t \in H$ in such a way that certain integrability conditions are satisfied.

The following observation is a foundation for the so called *mirror conjecture* for Calabi-Yau manifolds and its generalization to arbitrary symplectic manifolds suggested in [13]: Frobenius manifolds occur in the fields of mathematics quite remote from symplectic topology or enumerative geometry, and in particular — in singularity theory of isolated critical point of holomorphic functions. We outline below the *singularity theory – symplectic topology* dictionary.

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[1]Commutativity and symmetricity should be understood in the sense of super-algebra since the cohomology space is $\mathbb{Z}_2$-graded.

1. A germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ of holomorphic function at an isolated critical point of multiplicity μ.

2. Local algebra $\mathbb{C}[[z]]/(\partial f/\partial z)$.

3. Residue pairing $\langle \phi, \psi \rangle :=$ $\int_{|\partial f/\partial z_i| = \varepsilon_i} \frac{\phi(z)\psi(z)dz_1 \wedge ... \wedge dz_m}{(\partial f/\partial z_1)...(\partial f/\partial z_m)}$

4. Parameter space Λ of a miniversal deformation $f_\lambda(z)$ of the critical point; can be taken in the form $f_\lambda(z) :=$ $f(z) + \lambda_1 \phi_1 + ... + \lambda_\mu \phi_\mu$ where ϕ_α are to represent a basis in the local algebra.

5. Lagrangian submanifold $L \subset T^*\Lambda$ generated by f_λ, $L := \{(p, \lambda) | \exists z : d_z f_\lambda = 0, p = \frac{\partial f_\lambda}{\partial \lambda}\}$

6. Critical values $u_1, ..., u_\mu$ of Morse functions f_λ at the critical points.

7. Residue metric $\langle \partial_{\lambda_\alpha}, \partial_{\lambda_\beta} \rangle_\lambda :=$ $\sum_{z \in crit(f_\lambda)} \frac{(\partial_{\lambda_\alpha} f_\lambda) \ (\partial_{\lambda_\beta} f_\lambda)}{\det(\partial^2 f_\lambda/\partial z_i \partial z_j)}|_z$ diagonalized in the basis of non - degenerate critical points of a Morse function f_λ.

8. Hessians $\Delta := \det(\frac{\partial^2 f_\lambda}{\partial z_i \partial z_j})$ at the critical points.

1. A compact symplectic manifold X.

2. Cohomology algebra $H^*(X)$

3. Poincare pairing on $H^*(X)$, $\langle \phi, \psi \rangle := \int_X \phi \wedge \psi$.

4. The space $H := H^*(X)$ considered as a manifold.

5. Spectral variety $L \subset T^*H$ of the quantum cup-product $\circ_t$, $L \cap T_t^* H := Specm(T_t H, \circ_t)$. It is Lagrangian due to the integrability condition mentioned above.

6. Function $u : L \to \mathbb{C}$ such that $du = pdt|L$ considered as a multiple - valued function on H.

7. Poincare metric $\langle \phi, \psi \rangle_t = \sum_{p \in L \cap T_t^* H} \frac{\phi(p)\psi(p)}{\Delta(p)}$ diagonalized in the basis of idempotents of the quantum cup-product $\circ_t$ at semi-simple points t.

8. The function $\Delta : L \to \mathbb{C}$ *(quantum Euler class)* representing on $L \subset L \times_H L$ the cohomology class Poincare-dual to the diagonal in $X \times X$.

The objective of the present paper consists in extending the dictionary to include elliptic GW-invariants.

The 3-valent tensor $\langle a \circ_t b, c \rangle$ of structural constants of the quantum cup-product on $T_t H$ is actually defined by the formal series:

$$\langle a \circ_t b, c \rangle := \sum_{n=0}^{\infty} (a, b, c, t, ..., t)/n!$$

where the GW-invariant $(a, b, c, t, ..., t)$ counts the number of rational curves in X with $n + 3$ marked points situated on generic cycles whose homology classes are Poincare-dual respectively to $a, b, c, t, ..., t$.

The genus 1 GW-invariants in question can be similarly organized into a uni-valent tensor — an exact differential 1-form dG on H. The value of this 1-form on a tangent vector $a \in T_t H = H^*(X)$ is defined by the formal series

$$i_a dG := \sum_{n=0}^{\infty} [a, t, ..., t]/n!$$

where the GW-invariant $[a, t, ..., t]$ counts the number of elliptic curves in X with $n+1$ marked points situated on generic cycles representing respectively $a, t, ..., t$.

We propose the following construction for the singularity theory counterpart of the differential 1-form dG in Gromov-Witten theory. The critical values $u_1, ..., u_\mu$ of the functions f_λ can be taken on the role of local coordinates on the parameter space Λ of the miniversal deformation in the complement to the *caustic* — the critical value locus of the lagrangian map $L \subset T^*\Lambda \to \Lambda$. Consider the *complex oscillating integral*

$$I := \int_\Gamma e^{f_\lambda(z)/\hbar} v(z, \lambda) dz_1 \wedge ... \wedge dz_m.$$

The partial derivative $\hbar \partial I / \partial u_\alpha$ of the complex oscillating integral can be expanded into the stationary phase asymptotical series

$$\hbar^{m/2} e^{u_\alpha/\hbar} \frac{(\partial f_\lambda/\partial u_\alpha)|_{z_{crit}}}{\sqrt{\Delta_\alpha}} \left(1 + \hbar R_\alpha + o(\hbar)\right)$$

near the non-degenerate critical point z_{crit} corresponding to *the same* critical value u_α. The *asymptotical coefficients* R_α actually depend only on the 4-jet of f_λ and the 2-jet of v at z_{crit}. In terms of the critical values u_α, the

4

Hessians Δ_α and the asymptotical coefficients R_α the differential 1-form dG is described by the formula:

$$(*) \quad dG = \sum_{\alpha=1}^{\mu} (\frac{1}{48} d \log \Delta_\alpha + \frac{1}{2} R_\alpha du_\alpha).$$

Our proposal has implications in both singularity and Gromov-Witten theory.

In singularity theory, the residue metric on Λ (which is the counterpart of the flat Poincare metric on H) has no reason to be flat. However, according to K. Saito theory of primitive forms [25] one can choose the holomorphic volume form $v(z, \lambda) dz_1 \wedge ... \wedge dz_m$ (called *primitive*) in such a way that the corresponding residue metric is flat. Moreover, Saito's theory can be reformulated as the theorem that the above dictionary introduces a Frobenius structure on Λ *provided that* $(z_1, ..., z_m)$ everywhere in the dictionary means a unimodular coordinate system with respect to the primitive volume form. The same primitive form should be used in the definition of complex oscillating integrals involved into our construction of the 1-form $(*)$. With this hypothesis in force, we arrive to the following

Conjecture 0.1. *The* 1*-form (*) satisfies all axioms for the genus* 1 *GW-invariant dG.*

In particular, the conjecture suggests that the 1-form (*) satisfies the universal relation between genus 1 and genus 0 GW-invariants found by E. Getzler [10].

Remark on examples. We will see in Section 1 from the theory of Frobenius structures that differentials of the asymptotical coefficients R_α are expressible via the Hessians Δ_β by

$$dR_\alpha = \frac{1}{4} \sum_\beta (\partial_\alpha \log \Delta_\beta)(\partial_\beta \log \Delta_\alpha)(du_\beta - du_\alpha),$$

where ∂_γ means partial derivatives in the coordinate system $(u_1, ..., u_\mu)$. This allows to compute R_α if the Hessians are known as functions of all u_γ. However in applications to Gromov – Witten theory Δ_β are usually known only along some subspace in H, and the asymptotical coefficients are to be computed independently on the Hessians. All examples considered in this paper

are elementary and have $\mu = 2$. In such a case there are two coordinates u_γ which we usually denote $u_\pm$. The functions $R_\pm$ and $\Delta_\pm$ depend only on the difference $u = u_+ - u_-$. In the most examples we will have $R_\pm = \pm R(u)$ and $\Delta_\pm = \pm D(u)$. Then the formula $(*)$ reduces to

$$dG = \frac{1}{24} d\log\Delta(u) + \frac{1}{2}R(u)du, \text{ where } \frac{dR}{du} = \frac{1}{4}\left(\frac{d\log\Delta(u)}{du}\right)^2.$$

We use this method in the following example, but in some other examples we will present alternative techniques for computing asymptotical coefficients in order to illustrate computational tools available in applications to sympectic topology.

Example. The critical point $f = x^3$ of type A_2 has the miniversal deformation $x^3 - t_1 x + t_0$. At the critical point $x_\pm = \pm(t_1/3)^{1/2}$ we have the critical value $u_\pm = t_0 - 2x_\pm^3$ and the Hessian $\Delta_\pm = 6x_\pm$. Thus $\Delta(u) = (-u/4)^{1/3}$, $d(\log\Delta)/du = 1/3u$, $R(u) = \int du/(36u^2) = -1/(36u) + const$ where $const = 0$ by quasi-homogeneity. Therefore

$$dG = \frac{1}{24}\frac{du}{3u} - \frac{1}{2}\frac{du}{36u} = 0.$$

This result implies (by Hartogs principle) that for any isolated critical point dG defined by $(*)$ outside the caustic in the base of miniversal deformation extends holomorphicly to the whole base, and that $dG = 0$ for all simple singularities A_μ, D_μ, E_μ (since dG has zero quasi-homogeneity degree). The last conclusion agrees with E. Getzler's relation. [2]

In Gromov-Witten theory, the counterpart of complex oscillating integrals of singularity theory can be defined, as we shell see in the next section, entirely in terms of genus 0 GW-invariants. In particular, the coefficients R_α and the 1-form $(*)$ can be defined in intrincic terms of the Frobenius structure on H *provided* that the algebras $(T_t H, \circ_t)$ are semi-simple for generic $t \in H$.

Conjecture 0.2. *The elliptic GW-invariant dG of a compact symplectic manifold X with genericly semi-simple quantum cup-product is expressed by the formula $(*)$ in terms of rational GW-invariants.*

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[2]I am thankful to E. Getzler for correcting a mistake in my original computation.

In the rest of the paper we present some generalizations of the conjecture and our evidence in their favor. [3]

In Section 1 we review some definitions and results of genus 0 GW-theory, give a more precise formulation of Conjecture 0.2 and verify it directly in the example $X = \mathbb{C}P^1$.

In Section 2 we generalize the conjecture to the case of equivariant GW-theory on a Kähler manifold X provided with a Killing torus action with isolated fixed points only (toric and flag manifolds are main examples). The corresponding Frobenius structure is genericly semi-simple.

In Section 3 we prove the equivariant version of the conjecture.

In Section 4 we introduce and study GW-invariants of the non-compact manifolds which are total spaces of sums of negative line bundles over toric manifolds. Results of Sections 2 and 3 extend easily to such spaces. The key new point in this Section is the *mirror theorem* saying that those rational GW-invariants of such bundles which play the role of oscillating integrals in the Frobenius structure *are equal* to certain oscillating integrals defined in the spirit of toric hyper-geometric functions. In particular, the version of Conjecture 0.1 for such oscillating integrals holds true.

Results of Section 4 provide a new illustration to the so called mirror phenomenon: not only the GW-invariants of a manifold X form a structure *analogous* to the one observed in singularity theory, but the GW-invariants are explicitly *expressed* in terms of a specific datum of singularity theory type called the *mirror partner* of X.

In Section 5 we deal with toric *super*-manifolds which are objects dual to the bundles of Section 4 and whose GW-invariants are to coincide with GW-invariants of toric complete intersections. We invoke the nonlinear Serre duality theorem [12] (which relates genus 0 GW-invariants of a super-manifold and of the dual bundle space) in order to give a new proof of the mirror theorem for toric complete intersections [14]. The role of hyper-geometric functions is more transparent in this version of the proof. We believe that el-

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[3]While the paper was in preparation, B. Dubrovin and Y. Zhang [8] proved our conjectures 0.1 and 0.2. Namely, they showed that the 1-form (*) is the only *quasihomogeneous* solution to Getzler's relation mentioned above. I am thankful to Yu. Manin for this reference. Yet the results of our paper do not follow from [8] since we deal with equivariant generalizations of GW-theory where the quasihomogeneity properties of the 1-form (*) are relaxed (see Section 2). It is plausible however that the method of [8] can be extended to non-quasihomogeneous setting as well.

liptic GW-invariants of toric complete intersections are expressible in terms of the rational GW-invariants of the corresponding equivariant super-manifolds. However we are not ready to report on such applications because of the difficulties we explain in the end of Section 5.

I am thankful to organizers and participants of the summer-97 Taniguchi Symposium where the main results of this paper were first announced.

1   Gromov-Witten invariants and semi-simple Frobenius structures

We review here some basic properties of Gromov-Witten invariants of compact symplectic manifolds [20, 4, 3, 9, 21, 24, 26, 27, 29, 7].

Stable maps. Let (Σ, ε) denote a *prestable marked curve*, that is a compact connected complex curve Σ with at most double singular points and an ordered n-tuple $(\varepsilon_1, ..., \varepsilon_n)$ of distinct non-singular marked points. The *genus* of (Σ, ε) is defined as $g = \dim H^1(\Sigma, \mathcal{O}_\Sigma)$. The *degree* of a holomorphic map $f : (\Sigma, \varepsilon) \to X$ to a compact (almost) Kähler manifold X is defined as the total homology class $d \in H_2(X, \mathbb{Z})$ the map f represents. Two maps $f : (\Sigma, \varepsilon) \to X$ and $f' : (\Sigma', \varepsilon') \to X$ are called *equivalent* if there exist an isomorphism $\varphi : (\Sigma, \varepsilon) \to (\Sigma', \varepsilon')$ such that $f = f' \circ \varphi$. A holomorphic map $f : (\Sigma, \varepsilon) \to X$ is called *stable* if it has no non-trivial infinitesimal automorphisms. The set of equivalence classes of degree d stable holomorphic maps to X of genus g curves with n marked points is denoted $X_{g,n,d}$ (and called *moduli space of stable maps*). According to [3, 9, 21, 24] the moduli spaces have a natural structure of compact orbi-spaces, complex-analytic if X is Kähler. If $X = pt$, the spaces $X_{g,n,0}$ coincide with the Deligne-Mumford compactifications $\overline{\mathcal{M}}_{g,n}$ of moduli spaces of marked Riemann surfaces and are orbifolds of dimension $3g-3+n$ (unless empty, which happens for $g = 0, n < 3$ and $g = 1, n = 0$). For any X degree 0 stable maps form the moduli spaces $X_{g,n,0} = X \times \overline{\mathcal{M}}_{g,n}$.

One introduces the following tautological maps:

- *evaluation maps* $\mathrm{ev} = (\mathrm{ev}_1, ..., \mathrm{ev}_n) : X_{g,n,d} \to X^n$ defined by evaluating stable maps at the marked points;

- *forgetting maps* $\mathrm{ft}_i : X_{g,n+1,d} \to X_{g,n,d}, \ i = 1, ..., n$, well-defined (unless $d = 0$ and $\overline{\mathcal{M}}_{g,n}$ is empty) by forgetting the marked point ε_i followed by

contracting those irreducible components of Σ which have become unstable;

- *contraction maps* ct : $X_{g,n,d} \to \overline{\mathcal{M}}_{g,n}$ defined by forgetting the map $f : (\Sigma, \varepsilon) \to X$ followed by contracting unstable irreducible components of the marked curve (Σ, ε).

The diagram formed by the forgetting map $\mathrm{ft}_{n+1} : X_{g,n+1,d} \to X_{g,n,d}$ and by the evaluation map $\mathrm{ev}_{n+1} : X_{g,n+1,d} \to X$ is called the *universal stable map*: the fibre of ft_{n+1} over the point represented by a stable map $f : (\Sigma, \varepsilon) \to X$ is canonically identified with (the quotient of) the curve Σ (by the discrete group $Aut(f)$ of automorphisms of the map f if this group is non-trivial), and the restriction of ev_{n+1} to the fibre (lifted to (Σ, ε)) is equivalent to f. In particular, the sections $\varepsilon_1, ..., \varepsilon_n : X_{g,n,d} \to X_{g,n+1,d}$ defined by the marked points play the role of *universal* marked points on the universal stable map.

One introduces the *universal cotangent line* l_i which is a line (orbi-)bundle over $X_{g,n,d}$ with the fibre $T^*_{\varepsilon_i}\Sigma$ at the point $[f]$ and defined as the conormal bundle to the universal marked point ε_i. The 1-st Chern classes $c^{(1)}, ..., c^{(n)}$ of the orbi-bundles $l_1, ..., l_n$ are well defined over $\mathbb{Q}$.

Gromov − Witten invariants. Let $T(c) = t^{(0)} + t^{(1)}c + t^{(2)}c^2 + ...$ denote a formal power series with coefficients $t^{(i)}$ in the cohomology algebra [4] $H^*(X)$. Given n such series $T_1, ..., T_n$, one introduces the genus 0 Gromov-Witten invariant of X by

$$(T_1, ..., T_n)_d := \int_{[X_{0,n,d}]} (\mathrm{ev}_1^* T_1)(c^{(1)}) \wedge ... \wedge (\mathrm{ev}_n^* T_n)(c^{(n)}).$$

Here integration means evaluation of a cohomology class on the *virtual* fundamental class of the moduli space. If X is a *convex* Khler manifold, i. e. if the tangent bundle $\mathcal{T}_X$ is spanned by global vector fields on X, then the genus 0 moduli spaces $X_{0,n,d}$ are known to be compact complex orbifolds of complex dimension $\langle c_1(\mathcal{T}_X), d \rangle + \dim_{\mathbb{C}} X + n - 3$ (see [4]), and $[X_{0,n,d}]$ is the findamental class of the orbifold which is well-defined over $\mathbb{Q}$. In general the moduli spaces can have many irreducible components of different dimensions with nasty singularities. Newetherless one can endow them with rational virtual fundamental classes of *Riemann-Roch* dimension

$$\dim_{\mathbb{C}}[X_{g,n,d}] = \langle c_1(\mathcal{T}_X), d \rangle + (1 - g)(\dim_{\mathbb{C}} X - 3) + n$$

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[4]We always assume rational coefficients unless otherwise specified explicitly.

in such a way that the axioms [20] of Gromov-Witten theory are satisfied. We refer the reader to [3, 9, 21, 24] for several constructions of the virtual fundamental classes an for their properties. Using the classes $[X_{g,n,d}]$ one can introduce higher genus GW-invariants. [5] In this paper we will use the notation [6] $[T_1, ..., T_n]_d$ for the genus 1 GW-invariants of X. In the case when the series $T_i = t_i \in H^*(X)$ do not depend on c the GW-invariants $(t_1, ..., t_n)_d$ (resp. $[t_1, ..., t_n]_d$) have the enumerative meaning of the number of degree d rational (resp. elliptic) curves in X passing through n generic cycles Poincare-dual to $t_1, ..., t_n$.

We are going to use several universal identities between GW-invariants. [7]

The *WDVV-equation* says that the following sum is totally symmetric in A, B, C, D:

$$\sum_{n'+n''=n} \frac{1}{n'!n''!} \sum_{d'+d''=d} \sum_{\nu\nu'} (A, B, T, ..., T, \phi_\alpha)_{d'}^{n'+3} \eta^{\alpha\beta} (\phi_\beta, T, ..., T, C, D)_{d''}^{n''+3}.$$

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[5]More general GW-invariants (like $A(T_1, ..., T_n)_d$) corresponding to a choice of a cohomology class $A \in H^*(\overline{\mathcal{M}}_{g,n})$ are defined by adding the factor $\mathrm{ct}^* A$ to the integrand.

[6]We will add the super-script indicating the number of arguments as in $(T, ..., T)_d^n$ or $[T, ..., T]_d^n$ when the number would otherwise be ambiguous.

[7]While the identities are frequently used and their origin is well-known and explained for instance in [12], the actual proofs depend on details of the definition of the virtual fundamental cycles. A definition sufficient for our purposes is contained in [21]. It is based on the observation that the standard in algebraic geometry construction of the normal cone to the zero locus Z of an algebraic section of a vector bundle is intrincic with respect to the following data: (the variety Z, the complex $E \to F$ of vector bundles over Z defined by the linearization of the section). The kernel $T = ker(E \to F)$ is the algebraic tangent space to Z, and the cokernel N is called the obstruction space. The construction is adjusted to the orbi-bundle setting by applying it equivariantly on the total space of a suitable principal orbi-bundle. In the case when Z is a moduli space of stable maps the tangent and obstruction spaces are already defined (roughly, as the kernel and cokernel of the Cauchy - Riemann operator). One can explicitly point out a global resolution $T \to E \to F \to N$ of T and N by a suitable complex $E \to F$ of orbi-bundles. This defines the intrincic *normal subcone* in F, and the virtual fundamental cycle is defined as the intersection of this subcone with generic sections of F. As it is stated in [21], with this definition the standard arguments justifying the axioms [20] of GW-theory (they include the string, divisor, and WDVV-equations) go through. Detailed proofs are scattered however in a number of papers: see [2] for properties of GW-invariants, [11] for those of gravitational descendants, and [17] for equivariant generalizations of virtual fundumental classes including the localization formulas we will need in Sections 2 – 5.

10

Here $\{\phi_\alpha\}$ is a basis in $H^*(X)$, and $\sum_{\alpha\beta} \eta^{\alpha\beta} \phi_\alpha \otimes \phi_\beta$ represents the class in $H^*(X \times X)$ Poincare-dual to the diagonal. In particular the matrix $(\eta^{\alpha\beta})$ is inverse to the intersection matrix

$$\eta_{\alpha\beta} := \langle \phi_\alpha, \phi_\beta \rangle := \int_X \phi_\alpha \wedge \phi_\beta.$$

The *string* and *divisor* equations read respectively:

$$(1, T_1, ..., T_n)_d = \sum_{i=1}^n (T_1, ..., DT_i, ..., T_n)_d,$$

$$(p, T_1, ..., T_n)_d = \sum_{i=1}^n (T_1, ..., pDT_i, ..., T_n)_d + \langle p, d \rangle (T_1, ..., T_n)_d$$

where n should be at least 3 if $d = 0$, DT denotes the series $(T(c) - T(0))/c$, and $p \in H^2(X)$. The string and divisor equations hold true for genus 1 GW-invariants $[...]_d$ (with $n \geq 1$ if $d = 0$) and for GW-invariants of higher genus as well.

Gromov – Witten potentials. According to E. Witten [29] and B. Dubrovin [7], the WDVV-, string and divisor equations have several important interpretations in terms of the following generating functions for genus 0 GW-invariants. The genus 0 *GW-potential* is defined as the formal function of $t \in H^*(X)$:

$$F(t, q) = \sum_{n=0}^\infty \frac{1}{n!} \sum_{d \in \Lambda} q^d (t, ..., t)_d^n.$$

Here Λ denotes the *Mori cone* of X, the semi-group in the lattice $im \, (H_2(X, \mathbb{Z}) \to H_2(X, \mathbb{Q}))$ generated by those degrees of holomorphic curves in X for which the virtual fundamental classes $[X_{g,n,d}]$ are non-zero. All $d \in \Lambda$ have non-negative integer coordinates $(d_1, ..., d_r)$ with respect to a suitable integer basis $p_1, ..., p_r$ in $H^2(X, \mathbb{Q})$. Thus the formal completion $\mathbb{Q}[[\Lambda]]$ of the semigroup algebra of Λ is identified with a subalgebra in $\mathbb{Q}[[q_1, ..., q_r]]$. The symbol $q^d = q_1^{d_1}...q_r^{d_r}$ stands therefore for the element $d \in \Lambda$ in the semi-group algebra.

Denote t_0 the coordinate on $H^0(X)$, $(t_1, ..., t_r)$ — the coordinates on $H^2(X)$ with respect to the basis $\{p_i\}$, so that $t := \sum t_\alpha \phi_\alpha = t_0 + t_1 p_1 + ... + t_r p_r + ...$, and assume that the basis $\{\phi_\alpha\} = (1, p_1, ..., p_r, ...)$ is graded.

The GW-potential F has the following obvious properties:

- F is homogeneous of degree $3 - \dim_{\mathbb{C}} X$ with respect to the grading

$$\deg t_\alpha = 1 - \deg \phi_\alpha / 2, \ \deg q^d = \langle c_1(\mathcal{T}_X), d \rangle,$$

- $F(t, 0) = \int_X t \wedge t \wedge t / 6$,
- $\tilde{F}(t, q) := F(t, q) - F(t, 0)$ does not depend on t_0 and satisfies $q_i \partial \tilde{F} / \partial q_i = \partial \tilde{F} / \partial t_i$ (string and divisor equations for $(t, ..., t)_d$) and thus

$$\tilde{F}(t + \sum \tau_i p_i, q) = \tilde{F}(t, q \exp \tau).$$

We will also make use of the following generating functions:

$$S_{\alpha\beta}(t, q, \hbar) := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d \in \Lambda} q^d (\phi_\alpha, t, ..., t, \frac{\phi_\beta}{\hbar - c})_d^{n+2},$$

$$V_{\alpha\beta}(t, q, x, y) := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d \in \Lambda} q^d (\frac{\phi_\alpha}{x - c}, t, ..., t, \frac{\phi_\beta}{y - c})_d^{n+2}.$$

The ill-defined terms in these series are to be replaced as follows:

$$(\phi_\alpha, \frac{\phi_\beta}{\hbar - c})_0 := \eta_{\alpha\beta}, \ (\frac{\phi_\alpha}{x - c}, \frac{\phi_\beta}{y - c})_0 := \frac{\eta_{\alpha\beta}}{x + y}.$$

The tensor fields

$$\sum_{\varepsilon\varepsilon'} S_{\varepsilon\varepsilon'} dt_\varepsilon dt_{\varepsilon'} \text{ and } \sum_{\varepsilon\varepsilon'} V_{\varepsilon\varepsilon'} dt_\varepsilon dt_{\varepsilon'}$$

have degrees respectively $2 - \dim_{\mathbb{C}} X$ and $1 - \dim_{\mathbb{C}} X$ with respect to the above grading and $\deg \hbar = \deg x = \deg y = 1$.

In the following description of some identities between the GW-potentials F, S, V we will denote ∂_α the partial derivatives $\partial / \partial t_\alpha$ with respect to a basis $\{\phi_\alpha\}$ in $H^*(X)$. In the formulas below we will ignore the signs which may occur due to $\mathbb{Z}_2$-grading in cohomology and therefore assume that $H^*(X)$ has no odd part.

(1) Put $F_{\alpha\beta\gamma} := \partial_\alpha \partial_\beta \partial_\gamma F_0(t, q)$. *The* WDVV-equation for the GW-potential F reads:

$$\sum_{\varepsilon\varepsilon'} F_{\alpha\beta\varepsilon} \eta^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} \text{ is symmetric in } \alpha, \beta, \gamma, \delta \ .$$

This identity is interpreted as associativity of the *quantum cup-product* $\circ$: $H^*(X) \otimes H^*(X) \to H^*(X)$ defined by the structural constants

$$\langle \phi_\alpha \circ \phi_\beta, \phi_\gamma \rangle := F_{\alpha\beta\gamma}$$

(depending on parameters $t = \sum t_\alpha \phi_\alpha$ and $q = (q_1, ..., q_r)$). The quantum cup-product is commutative, symmetric relatively the intersection form,

$$\langle \phi_\alpha \circ \phi_\beta, \phi_\gamma \rangle = \langle \phi_\alpha, \phi_\beta \circ \phi_\gamma \rangle ,$$

and the unity $1 \in H^*(X)$ remains the unity for the quantum cup-product.

(2) The WDVV-equation for F is also interpreted as integrability of the following system of linear PDE for a vector-function of t (depending also on the parameters q and $\hbar$) with values in the cohomology space of X:

$$(**) \quad \hbar \partial_\alpha \vec{s} = \phi_\alpha \circ \vec{s} .$$

The formally adjoint system with respect to the intersection form is $-\hbar \partial_\alpha \vec{s} = \phi_\alpha \circ \vec{s}$:

$$\forall \alpha \quad \partial_\alpha \langle \vec{s}(t, q, \hbar), \vec{s}(t, q, -\hbar) \rangle = 0.$$

(3) The generating functions $(S_{\beta\gamma})$ form a fundamental solution matrix S for the system of PDE:

$$\hbar \partial_\alpha S_{\beta\gamma} = \sum_{\varepsilon\varepsilon'} F_{\alpha\beta\varepsilon} \eta^{\varepsilon\varepsilon'} S_{\varepsilon'\gamma} .$$

Namely, WDVV-equations imply $\phi_\alpha \circ \partial_\beta S = \phi_\beta \circ \partial_\alpha S$ while the string equation implies that $\hbar \partial_0 S = S$.

(4) Application of WDVV-equations implies

$$\partial_0 V_{\alpha\beta}(t, q, x, y) = \sum_{\varepsilon\varepsilon'} \partial_0 S_{\varepsilon\alpha}(t, q, x) \eta^{\varepsilon\varepsilon'} \partial_0 S_{\varepsilon'\beta}(t, q, y).$$

Together with the string equation this yields the *unitarity condition*

$$\sum_{\varepsilon\varepsilon'} S_{\varepsilon\alpha}(t, q, \hbar) \eta^{\varepsilon\varepsilon'} S_{\varepsilon'\beta}(t, q, -\hbar) = \eta_{\alpha\beta}$$

and the relation

$$V_{\alpha\beta}(t, q, x, y) = \frac{1}{x+y} \sum_{\varepsilon\varepsilon'} \eta^{\varepsilon\varepsilon'} S_{\varepsilon\alpha}(t, q, x) S_{\varepsilon'\beta}(t, q, y).$$

(5) The divisor equation with $p = \sum \tau_i p_i$ applied to $S(t, q, \hbar)$ shows that

$$S_{\alpha\beta}(t + p, q, \hbar) = \sum_{\varepsilon} S_{\alpha\varepsilon}(t, qe^\tau, \hbar) \eta^{\varepsilon\varepsilon'} \langle \phi_{\varepsilon'} e^{p\tau/\hbar}, \phi_\beta \rangle.$$

This property together with the unitarity condition and the asymptotics $S_{\alpha\beta}|_{t=0,q=0} = \eta_{\alpha\beta}$ uniquely specifies S among fundamental solutions of the differential system $(**)$.

Frobenius manifolds. According to B. Dubrovin [7], genus 0 GW-invariants of X define on the cohomology space of X the structure of a Frobenius manifold. We review here Dubrovin's theory of canonical coordinates on semi-simple Frobenius manifolds.

A *Frobenius algebra* structure on a vector space consists of a commutative associative multiplication $\circ$ with unity 1 and a linear function α such that $\langle u, v \rangle := \alpha(u \circ v)$ is a non-degenerate bilinear form.

A *Frobenius structure* on a manifold H is a field of Frobenius algebra structures on the tangent spaces $T_t H$ satisfying the following integrability conditions:

(a) the metric $\langle \cdot, \cdot \rangle$ is flat: $\nabla^2 = 0$,

(b) the unity vector field 1 is covariantly constant: $\nabla 1 = 0$,

(c) the 1-st order linear PDE system for sections s of TH defined by $\hbar \nabla_w s = w \circ s$ is consistent for any $\hbar \neq 0$.

The Frobenius manifold is said *conformal of dimension $D \in \mathbb{Q}$* if it is provided with a vector field E (called *Euler*) such that the tensor fields $1, \circ$ and $\langle \cdot, \cdot \rangle$ are eigen-vectors of the Lie derivative operator L_E with the eigen-values respectively $-1, 1$ and $2 - D$.

In flat coordinates $\{t_\alpha\}$ of the metric the condition (c) can be reformulated as flatness for any $\hbar$ of the connection

$$(1) \quad \nabla_\hbar := \hbar d - \sum_\alpha A_\alpha(t) dt_\alpha \wedge$$

where A_α are the multiplication operators $\partial_\alpha \circ_t$.

Using the property of the structural constants $F_{\alpha\beta\gamma}$ of the quantum cup-product on $H^*(X)$ to depend on $q_i, t_i, i = 1, ..., r$ only in the combinations $q_i \exp t_i$, we see that the quantum cup-product and the Poincare pairing define on $H = H^*(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})$ the structure of a (formal) Frobenius

manifold of conformal dimension $D = \dim_{\mathbb{C}} X$ with respect to the Euler vector field

$$E = t_0 \partial_0 + \sum_{i=1}^{r} c_i \partial_i + \sum_{\alpha : \deg t_\alpha < 0} \deg(t_\alpha) t_\alpha \partial_\alpha.$$

Here $\sum c_i p_i$ is the 1-st Chern class of $\mathcal{T}_X$.

Given a pensil of flat connections $\nabla_\hbar$ one can study asymptotical behavior of horizontal sections as $\hbar \to 0$. The asymptotics is described by the following data.

- The characteristic Lagrangian variety $L \subset T^*H$ defined as the spectrum $Spec(Vect(H), \circ)$ of the algebra of vector fields on H with the multiplication $\circ$. Flatness of $\nabla_\hbar = \hbar d - A^1$ is equivalent to $A^1 \wedge A^1 = 0$ and $dA^1 = 0$. The first condition means commutativity $[A_\alpha, A_\beta] = 0$ while the second one *implies* that L is Lagrangian at generic points [16].

- The function u on L, may be multiple-valued, defined as a potential for the action 1-form $\sum p_\alpha dt_\alpha$ on T^*H restricted to L. In our case of conformal frobenius structures u can be chosen as the restriction to L of the function $\sum c_i p_i + \sum (\deg t_\alpha) p_\alpha t_\alpha$ on T^*H defined by the Euler vector field E.

- The function Δ on L defined by the metric $\langle p, p \rangle$ on T^*H. It is the restriction to L of $\sum p_\alpha \eta^{\alpha\beta} p_\beta$.

A point $t \in H$ where the algebra $(T_t H, \circ_t)$ is semi-simple is called *semi-simple*. In a neighborhood of a semi-simple point the characteristic variety L consists of $N = \dim H$ sections of T^*H which span each $T_t^* H$ so that the corresponding branches $u_\alpha, \alpha = 1, ..., N$, of the function u form a local coordinate system on H called *canonical*. The vector fields $f_\alpha = \Delta^{-1/2} \partial/\partial u_\alpha$ form an orthonormal basis diagonalising $\circ$. Let Ψ denote the transition matrix from the basis $\partial/\partial t_\alpha$ to f_β: $\partial/\partial t_\alpha = \sum \psi_{\alpha\beta} f_\beta$.

Proposition 1.1 (see [7].) *In a neighborhood of a semi-simple point there exists a fundamental solution of the system $\nabla_\hbar s = 0$ represented by the asymptotical series*

$$(2) \quad \Psi(1 + \hbar R^{(0)} + \hbar^2 R^{(1)} + ...) \exp(U/\hbar)$$

where $U = diag(u_1, ..., u_N)$ is the diagonal matrix of canonical coordinates.

Proof. Indeed, substituting the series into the equation we obtain the chain of equations

$$A^1 \Psi = \Psi dU, \ \Psi^{-1} d\Psi = [dU, R^{(0)}],$$

15

$$DR^{(0)} = [dU, R^{(1)}], \ ..., \ DR^{(k)} = [dU, R^{(k+1)}], ...$$

where the connection operator $D = d + \Psi^{-1}d\Psi\wedge$ is flat and anti- commutes with $dU\wedge$. The first equation means that Ψ diagonalizes A^1 to dU. Columns of Ψ form an orthogonal basis since A^1 is self-adjoint with respect to $\langle \cdot, \cdot \rangle$. The next equation requires the columns to be normalized to constant lengths and expresses off-diagonal entries of $R^{(0)}$ via Ψ. In particular $R^{(0)}$ is symmetric. [8] The diagonal entries of $R^{(0)}$ can be found by integration from the next equation:

$$dR_{ii}^{(0)} = \sum_l R_{il}^{(0)}(du_l - du_i)R_{li}^{(0)}.$$

Closedness of the RHS is easy to derive directly from the flatness $D^2 = 0$. Continuing the inductive procedure, we express the off-diagonal part of $R^{(k+1)}$ via $R^{(k)}$ algebraicly from $[dU, R^{(k+1)}] = DR^{(k)}$, and find the diagonal part of $R^{(k+1)}$ by integration from the next equation. Let us check compatibility conditions.

First, $DR^{(k)}$ has the zero diagonal (induction hypothesis) and thus is a commutator with dU due to De Rham lemma: the anti-commutator $\{dU, DR^{(k)}\} = D[R^{(k)}, dU] = -D^2R^{(k-1)} = 0$.

It remains to verify exactness of $\sum_l R_{il}^{(0)}(du_l - du_i)R_{li}^{(k+1)}$. It can be reformulated as $d(R^{(0)}[dU, R^{(k+1)}])_{diag} = 0$. We have:

$$d(R^{(0)}[dU, R^{(k+1)}]) = (dR^{(0)}) \wedge DR^{(k)} + R^{(0)}dDR^{(k)} =$$

$$(dR^{(0)} - [dU, R^{(0)}]) \wedge DR^{(k)} = (DR^{(0)})^t \wedge DR^{(k)} = [(R^{(1)})^t, dU] \wedge [dU, R^{(k+1)}]$$

which has zero diagonal entries. $\square$

Remarks. (1) The asymptotical solution of Proposition 1.1 is not unique. First, the canonical coordinates are defined up to a constant summand. When the choice has been made, the matrix Ψ of eigen-vectors is defined up to the right multiplication by a constant diagonal matrix. Such a multiplication conjugates all $R^{(k)}$ by this matrix and thus does not change the diagonal entries. Finally, another choice of intgration constants for $R_{ii}^{(k)}$ gives rise to the right multiplication of the whole series by a diagonal matrix $diag\ (C_1(\hbar), ..., C_N(\hbar))$, where $C_i = 1 + c_i^{(0)}\hbar + c_i^{(1)}\hbar^2 + ...$ and $c_i^{(j)}$ are constants.

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[8]For those familiar with Dubrovin's notations: our $R_{ij}^{(0)}$, $i \neq j$, coincide with his γ_{ij}.

(2) In the canonical coordinate system $(u_1, ..., u_N)$ flatness of the connection D reads: [9]

$$\partial_l R_{ij}^{(0)} = R_{il}^{(0)} R_{lj}^{(0)} \ , l \neq i, j, \ \sum_l \partial_l R_{ij}^{(0)} = 0.$$

Since $R^{(0)}$ is symmetric, this implies that the 1-form

$$dR := \sum_{i=1}^{N} R_{ii}^{(0)} du_i$$

is closed.

(3) The flat metric on H is diagonal in the canonical coordinate system and takes on $\sum_i \Delta_i^{-1} (du_i)^2$. The connection D is the Levi-Civita connection of this metric written in the basis of vector fields $\Delta_i^{1/2} \partial/\partial u_i$. Thus the coefficients R_{ij} with $i \neq j$ can be computed in terms of the metric. This observation leads to the formula mentioned in the introduction:

$$dR_{ii}^{(0)} = \frac{1}{4} \sum_j (\partial_i \log \Delta_j)(\partial_j \log \Delta_i)(du_j - du_i).$$

(4) In the conformal case the Euler field assumes in the canonical coordinates the form $E = \sum u_l \partial_l$. The homogeneity relation $L_E R^{(0)} = -R^{(0)}$ together with the above flatness condition form a remarkable system of Hamiltonian differential equations which determines $R^{(0)}$. Namely, following B. Dubrovin [7] consider the anti-symmetric matrix $V_{ij} := R_{ij}^{(0)}(u_i - u_j)$ as a point in the Poisson manifold so_N^*. Introduce N non-autonomous quadratic hamiltonians

$$H_i := \frac{1}{2} L_E R_{ii}^{(0)} = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij} V_{ji}}{u_i - u_j}.$$

These hamiltonians Poisson-commute on so_N^* and their flows determine the dependence of V on u: $\partial_i V_{jl} = \{H_i, V_{jl}\}$. [10]

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[9]The last relation means that the vector $\sum \partial_l$ represents the unity in $T_t H$ and holds true for all $R_{ij}^{(k)}$.

[10]In fact $H_i = -R_{ii}^{(0)}/2 + const$ in the case of conformal Frobenius structures. I am thankful to B. Dubrovin for this observation. The last relation fails however in the more general setting we will encounter in the next section.

Elliptic GW-invariants. Introduce the *elliptic* GW-potential of X

$$G(t, q) := \sum_{d \in \Lambda} \sum_{n=0}^{\infty} q^d [t, ..., t]_n^d / n!.$$

The degree 0 part of G equals

$$G(t, 0) = -\frac{1}{24} \int_X t \wedge c_{\dim X - 1}(T_X),$$

while the non-zero degree terms depend on q_i only in the combinations $q_i \exp t_i$ due to the divisor equation. Thus dG can be considered as a closed 1-form on the Frobenius manifold $H = H^*(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})$. It has homogeneity degree 0 with respect to the Euler vector field on H.

Let $t \in H$ be a semi-simple point. In a neighborhood of t the functions $\log \Delta_\alpha$, $R_{\alpha\alpha}^{(0)}$ are uniquely defined up to additive constants. We normalize the constants in $R_{\alpha\alpha}^{(0)}$ by the homogeneity condition

$$L_E R_{\alpha\alpha}^{(0)} = -R_{\alpha\alpha}^{(0)}.$$

Conjecture 1.2. *Suppose that the Frobenius structure defined by the rational GW-potential F of the manifold X is semi-simple. Then the elliptic GW-potential of X is determined by*

$$dG = \sum_\alpha d(\log \Delta_\alpha)/48 + \sum_\alpha R_{\alpha\alpha}^{(0)} du_\alpha / 2.$$

Example: $X = \mathbb{C}P^1$. The classes 1 and p Poincare-dual to the fundamental class and a point form a basis in $H^*(X)$. The elliptic GW-potential is $G = -t/24$ where t is the coordinate on $H^2(X)$ since non-constant elliptic curves do not contribute to G for dimensional reasons. Our conjecture agrees with this fact. Indeed, looking for the fundamental solution $S = \Psi(1 + R\hbar + o(\hbar)) \exp(U/\hbar)$ of the differential system $\hbar \dot{S}_1 = S_2$, $\hbar \dot{S}_2 = e^t S_1$ corresponding to the quantum cohomology algebra $\mathbb{Q}[p, q]/(p^2 - q)$ of $\mathbb{C}P^1$, we find

$$\Psi^{-1} \dot{\Psi} = [\dot{U}, R], \quad (\dot{R} + \Psi^{-1} \dot{\Psi} R)_{diag} = 0.$$

The normalized eigen-vectors of the quantum multiplication operator $p\circ$ corresponding to the eigen-values $\dot{u}_\pm = \pm e^{t/2}$ are equal to $(e^{-t/4}p \pm e^{t/4})$. We find $(\Psi^{-1}\dot{\Psi})_{+-} = -1/4$ and $R_{+-} = e^{-t/2}/8 = -R_{-+}$. Respectively,

$$\dot{R}_{++} = -R_{+-}(\dot{u}_+ - \dot{u}_-)R_{-+} = \exp(-t/2)/32 = -\dot{R}_{--}$$

and therefore $R_{++}du_+ + R_{--}du_- = -dt/8$.

On the other hand, $\Delta_\pm = \pm 2e^{t/2}$ and thus $d\log\Delta_+\Delta_- = dt$. We find $dG = dt/48 - dt/16 = -dt/24$.

2 Equivariant GW-invariants in genus 0 and 1

In this section we formulate a theorem confirming an equivariant version of Conjecture 1.2 in the case of toric actions with isolated fixed points.

Equivariant cohomology. Let a compact group G act on a topological space M. The *equivariant* cohomology $H_G^*(M)$ is defined as the cohomology of the homotopy quotient $M_G := (EG\times M)/G$ where EG is the total space of the universal principal G-bundle $EG \to BG$. When M is a point $M_G = BG$, and the ring $H_G^*(pt) = H^*(BG)$ plays the role of the coefficient ring in the G-equivariant cohomology theory. In particular, the G-equvariant map $M \to pt$ induces the M-bundle $M_G \to BG$ and a natural structure of the $H_G^*(pt)$-module on $H_G^*(M)$.

A G-equivariant vector bundle V over a G-space M induces a vector bundle V_G over the homotopy quotient M_G. Equivariant characteristic classes of V are defined as the ordinary characteristic classes of V_G. This construction applies to equivariant orbi-bundles over orbi-spaces and gives rise to equivariant characteristic classes of orbi-bunles well-defined in $H_G^*(M, \mathbb{Q})$.

In the case of smooth orientation-preserving G-actions on compact oriented manifolds the *fiberwise integration* over the fibres of the M-bundle $M_G \to BG$ defines the $H_G^*(pt)$-linear homomorphism $\int_M : H_G^*(M) \to H_G^*(pt)$ and the bilinear Poincaré pairing

$$\langle \phi, \psi \rangle := \int_{[M]} \phi \wedge \psi,$$

non-degenerate over $H^*(BG, \mathbb{Q})$ in the case of Hamiltonian G-actions on compact symplectic manifolds. The same operations are well-defined over $\mathbb{Q}$ in the case of orbifolds.

Let us assume now that G is a torus. According to the Borel fixed point localization formula

$$\int_{[M]} \phi = \int_{[M^G]} \frac{i^*\phi}{Euler_G(N_M(M_G))}$$

where $i : M^G \to M$ is the inclusion of the fixed point submanifold M^G into M, $N_M(M^G)$ is the normal bundle to the fixed point submanifold, and $Euler_G$ is the G-equivariant Euler class. In the orbifold case $N_M M^G$ is an equivariant orbi-bundle over the orbifold M^G, and the localization formula holds true over $\mathbb{Q}$. One should have in mind however that the fundamental class of the fixed point *sub*-orbifold M^G differs from the geometrical fundamental class of the orbifold M^G by the factors $1/|Aut|$ on each connected component, where Aut is the subgroup — in the symmetry group defining the orbifold structure on M at a generic point of the component — stabilizing the point.

Genus 0. Let X be a compact Kähler manifold provided with a Hamiltonian Killing action of a compact Lie group G. Then the group acts also on the moduli spaces of stable maps $X_{g,n,d}$, and this action commutes with evaluation, forgetting and contraction maps. The constructions [3, 21, 24] of the virtual fundamental cycles $[X_{g,n,d}]$ can be extended to the equivariant setting. [11] This allows one to generalize GW-theory to the equivariant case. The theory of equivariant genus 0 GW-invariants [12] is quite analogous to the theory of Frobenius structures reviewed in Section 1. We describe below the modifications to be made in the equivariant setting emphasizing the case of tori actions.

(1) The coefficient algebra $H^*_G(pt, \mathbb{Q})$ of the equivariant cohomology theory replaces the ground field $\mathbb{Q}$ of the non-equivariant GW-theory. If G is the l-dimensional torus the algebra is isomorphic to the polynomial ring $\mathbb{Q}[\lambda_1, ..., \lambda_l]$ in l generators of degree 1 (in our complex grading) since BG is weakly homotopy equivalent to $(\mathbb{C}P^\infty)^l$. In all questions involving Borel

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[11]For convex X the G-equivariant virtual fundamental class $[X_{0,n,d}]$ coincides with the equivariant fundamental class of $X_{0,n,d}$ considered as an orbifold.

localization formulas the algebra is replaced by the field of fractions $\mathbb{Q}(\lambda)$ since rational functions of λ can occur.

(2) The equivariant GW-invariants $(T_1, ..., T_n)_n^d$ and their higher genus counterparts take values in $\mathbb{Q}[\lambda]$ and are polylinear over $\mathbb{Q}[\lambda]$. The potentials $F(t, q), S_{\alpha\beta}(t, q, \hbar), V_{\alpha\beta}(t, q, x, y)$, etc., can be therefore considered as formal functions with coefficients in $\mathbb{Q}[\lambda]$ and thus are functions of λ as well. Note that $t = \sum t_\alpha \phi_\alpha$ is now the general *equivariant* cohomology class of X, and that $\{\phi_\alpha\}$ here represents a $\mathbb{Q}(\lambda)$-basis in the $\mathbb{Q}[\lambda]$-module $H_G^*(X)$.

(3) The string equation remains unchanged in the equivariant case. Assuming for simplicity that X is simply connected (which is automatically the case if the Hamiltonian torus action has only isolated fixed points) we have the short exact sequence

$$0 \rightarrow H^2(BG) \rightarrow H_G^2(X) \rightarrow H^2(X) \rightarrow 0.$$

The divisor equation and its consequences for GW-potentials hold true in the equivariant case as well if only we interpret $\langle p, d \rangle$ as the value of the projection of $p \in H_G^2(X)$ to $H^2(X)$ on the homology class $d \in H_2(X)$. Nevertheless it will be convenient sometimes to keep the formal variables $q_1, ..., q_r$ in place and thus to consider $(\mathbb{Q}[\lambda])[[\Lambda]]$ as the ground algebra in the equivariant setting.

(4) The equivariant GW-potential F defines on $H := H_G^*(X, \mathbb{Q}[[\Lambda]])$ a Frobenius structure over the ground ring $(\mathbb{Q}[\lambda])[[\Lambda]]$. The divisor equation induces however the following symmetry:

$$(\partial_i - q_i \partial/\partial q_i)\tilde{F}_{\alpha\beta\gamma}(t, q, \lambda) = 0, \quad i = 1, ..., r,$$

where ∂_i are the tangent vector fields along $H_G^2(X)$ representing the basis in $H^2(X)$ as in Section 1. The grading axiom should be modified in the equivariant case: the potential F is homogeneous of degree $3 - \dim X$ with respect to the Euler field

$$E = \sum_\alpha (\deg t_\alpha) t_\alpha \partial_\alpha + \sum_i c_i q_i \partial/\partial q_i + \sum_j \lambda_j \partial/\partial \lambda_j.$$

Due to the last summand the Euler derivation is not $\mathbb{Q}[\lambda]$-linear, and thus the axioms of the conformal Frobenius manifold are not satisfied. We will call the Frobenius structures with such a modified grading axiom *quasi-conformal.*

Let us assume now that the action of the torus G on X has only isolated fixed points. The δ-functions of the fixed points form a basis $\{\phi_\alpha\}$ in the equivariant cohomology of X over the field of fractions $\mathbb{Q}(\lambda)$. The classical equivariant cohomology algebra of X is semi-simple at generic λ and therefore its quantum deformation and the corresponding quasi-conformal Frobenius manifold H is genericly semi-simple as well. Thus Proposition 1.1 applies and gives rise to the expansion [12]

$$(S_{\alpha\beta}(t,q,\hbar,\lambda)) = \Psi(1 + \hbar R^{(0)} + o(\hbar))\exp(U/\hbar) \ .$$

The canonical coordinates u_α and the diagonal entries $R^{(0)}_{\alpha\alpha}$ of the matrix $R^{(0)}$ are defined by the Frobenius structure up to additive "constants" which are now elements of the ground ring $(\mathbb{Q}[\lambda])[\Lambda]$. [13] Moreover, the symmetry induced by the divisor equation for the potentials $S_{\alpha\beta}$ implies that $R^{(0)}$ is invariant with respect to the vector fields $\partial_i - q_i\partial/\partial q_i$. This allows us to normalize the additive constants by the condition $R^{(0)}_{\alpha\alpha} \equiv 0 \mod (q)$.

Genus 1. Introduce now the equivariant genus 1 potential

$$G(t,q,\lambda) := \sum_{d\in\Lambda}\sum_{n=0}^{\infty} q^d[t,...,t]^d_n/n!.$$

The degree $d = 0$ part of G can be computed by the localization formulas:

$$G(t,0,\lambda) = -\frac{1}{24}\int_X t \wedge c_{\dim X-1}(\mathcal{T}_X) = -\frac{1}{24}\sum_\alpha t_\alpha c^\alpha_{-1},$$

where c^α_{-1} is defined to be the ratio $c_{\dim X-1}(\mathcal{T}_X)/c_{\dim X}(\mathcal{T}_X)$ of the equivariant Chern classes localized to the fixed point $\alpha \in X^G$.

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[12]We will discuss it with greater detail in Section 3 in connection with localization formulas.

[13]Dubrovin's classification of semi-simple Frobenius structures should be modified in the quasi-conformal case as explained in [12]. In particular, the Hamiltonians H_i play now the role of densities in the Poisson-commuting Hamiltonians on the affine Lie coalgebra $\hat{so}^*_N$. They can be also described as the Lie derivatives $H_i = \sum_\alpha u_\alpha(\partial/\partial u_\alpha)R^{(0)}_{ii}$ but are no longer proportional to $R^{(0)}_{ii}$ since these functions are quasi-homogeneous with respect to the Euler field E which takes on $\sum u_\alpha\partial/\partial u_\alpha + \sum \lambda_j\partial/\partial\lambda_j + \sum c_i q_i\partial/\partial q_i$ in the canonical coordinate system.

Theorem 2.1. *Suppose that the complexified action of the torus $G_{\mathbb{C}}$ on the compact Kähler manifold X has only isolated fixed points and isolated 1-dimensional orbits. Then*

$$dG = \sum_\alpha d(\log \Delta_\alpha)/48 - \sum_\alpha c^\alpha_{-1} du_\alpha/24 + \sum_\alpha R^{(0)}_{\alpha\alpha} du_\alpha/2.$$

Remarks. (1) The differential d in the theorem is taken with respect to the coordinates t_α on the space H over the ground ring $(\mathbb{Q}[\lambda])[\Lambda]$. Thus q and λ are considered as constants.

(2) Redefining the additive constants in $R^{(0)}_{\alpha\alpha}$ by

$$R_\alpha := R^{(0)}_{\alpha\alpha} - c^\alpha_1/12$$

we can reformulate the theorem in the form

$$dG = \sum d \log \Delta_\alpha/48 + \sum R_\alpha du_a/2$$

suggested in the introduction. In the several examples we tried both summands in the RHS have limits as λ approaches 0 and in this limit turn into their non-equivariant counterparts. If proven to be the general rule, this observation would confirm the Conjecture 1.2 for toric manifolds and homogeneous Kähler spaces.

Example. Equivariant quantum cohomology of $\mathbb{C}P^1 = P(\mathbb{C}^2)$ with respect to the circle acting by $diag(e^{i\varphi}, e^{-i\varphi})$ on $\mathbb{C}^2$ is known to be isomorphic [14] to $\mathbb{Q}[p, q, \lambda]/(p^2 - \lambda^2 - q)$ with the equivariant Poincaré pairing

$$\langle \phi, \psi \rangle = \frac{1}{2\pi i} \oint \phi(p, q, \lambda)\psi(p, q, \lambda) \frac{dp}{p^2 - \lambda^2 - q} \; .$$

Introducing the indices $\pm$ for the two fixed points on $\mathbb{C}P^1$ with the normal Euler classes $\pm 2\lambda$, we find that the normalized "Hessians" $\Delta_\pm$ are equal to $\pm 2p/(\pm 2\lambda)$ where $p = (1 + q/\lambda^2)^{1/2} = \lambda + O(q)$. The differentials of the canonical coordinates are given by

$$du_\pm = \pm p\, d \log q = \pm \frac{2p^2 dp}{p^2 - \lambda^2} \; .$$

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[14]We write here q instead of $q \exp t$ in order to minimize the paperwork.

In the basis $\phi_\pm = (\lambda \pm p)/2\lambda$ in $H_G^*(\mathbb{C}P^1)$ the matrix Ψ of normalized eigenvectors of quantum cup-product operators takes on

$$\Psi = \frac{1}{2}\left[\begin{array}{cc} z + z^{-1} & -z + z^{-1} \\ z - z^{-1} & -z - z^{-1} \end{array}\right] ,$$

where $z := (1 + q/\lambda^2)^{1/4} = 1 + \dots$. Respectively,

$$\Psi^{-1}\dot{\Psi} = \left[\begin{array}{cc} 0 & -\dot{z}/z \\ -\dot{z}/z & 0 \end{array}\right] ,$$

where the dot means qd/dq.

The diagonal entries of $R^{(0)}$ can be found by integration of $\pm(\dot{z})^2/z^2(\dot{u}_+ - \dot{u}_-) = \pm(p^2 - \lambda^2)/32p^5$. Since $d\log q = (2p/(p^2 - \lambda^2))dp$, we have

$$R_{++}^{(0)} = -R_{--}^{(0)} = \int_\lambda^p \frac{x^2 - \lambda^2}{16x^4}dx = -\frac{1}{16p} + \frac{\lambda^2}{48p^3} + \frac{1}{24\lambda} .$$

The contribution to $\dot{G}$ via Theorem 2.1 equals

$$\frac{1}{2}(R_{++}^{(0)}\dot{u}_+ + R_{--}^{(0)}\dot{u}_-) = -\frac{1}{16} + \frac{\lambda^2}{48p^2} + \frac{p}{24\lambda} .$$

The other two summands are

$$-\frac{1}{24}\left(\frac{\dot{u}_+}{2\lambda} + \frac{\dot{u}_-}{-2\lambda}\right) = -\frac{p}{24}$$

(which together with the first one yields $-1/16 + \lambda^2/48p^2$) and

$$2\frac{\dot{\Delta}}{24\Delta} = \frac{\dot{p}}{24p} = \frac{p^2 - \lambda^2}{48p^2} .$$

The total sum $-1/16 + 1/48 = -1/24$ agrees with the known result $G = -(\log q)/24$.

3   Fixed point localization in genus 0 and 1

The proof of Theorem 2.1 is based on application of the Borel fixed point localization formula to the equivariant virtual fundamental classes $[X_{g,n,d}]$

of the moduli spaces of stable maps. A stable map $f : (\Sigma, \varepsilon) \to X$ represents a fixed point of the torus action in the moduli space if its shift by the torus action can be compensated by automorphisms of the marked curve. Equivalently,

- $f(\Sigma)$ is contained in the union of 0- and 1-dimensional orbits of $G_{\mathbb{C}}$,
- $f(\varepsilon)$ is contained in the fixed point set X^G,
- if the map f restricted to an irreducible component of Σ is constant then the image is a fixed point,
- if it is not constant then the component is isomorphic to $\mathbb{C}P^1$, carries no more than 2 special points (which can be positioned at 0 and/or ∞, and the map is a multiple cover $z \mapsto w = z^m$ onto the closure of a 1-dimensional orbit (which is also isomorphic to $\mathbb{C}P^1$ with $w = 0, \infty$ to be the fixed points).

The connected component of the fixed point set $X^G_{g,n,d}$ containing the equivalence class $[f]$ can be described as the product of the Deligne-Mumford spaces $\overline{\mathcal{M}}_{g_i,k_i}$ factorized by a finite symmetry group of the combinatorial structure of the map f. Each factor $\overline{\mathcal{M}}_{g_i,k_i}$ corresponds to a connected component Σ_i in $f^{-1}(X^G)$, g_i is the arithmetical genus of Σ_i, and k_i equals the total number of marked points and of special points $\overline{(\Sigma - f^{-1}(X^G))} \cap f^{-1}(X^G)$ situated on Σ_i.

Application of the fixed point localization formula requires a description of the virtual normal bundle to each fixed point component and reduces to integration over the Deligne-Mumford spaces. The idea to apply the localization technique to the moduli spaces $X_{g,n,d}$ is due to M. Kontsevich [19] and was systematically exploited in [12, 14] and several other papers. The description used in [19, 12, 13] for localizations of the fundamental classes $[X_{0,n,d}]$, being obvious in the orbifold case, can be easily extended to the general "virtual" case. A rigorous justification of these localization formulas was recently given in [17] on the basis of the algebraic-geometrical approach [3] to the virtual fundamental cycles.

The idea of our proof of Theorem 2.1 can be now described as follows. Any fixed point of the torus G action on the genus 1 moduli spaces $X_{1,n,d}$ has the following combinatorial structure:

- either it is a tree walking along the skeleton of 1-dimensional orbits in X, and $f^{-1}(X^G)$ has exactly one connected component Σ_0 of arithmetical genus 1,
- or it is a graph walking along the skeleton of 1-dimensional orbits with

exactly one cycle, and all irreducible components are rational.

The way how the fixed points of the first type contribute to the genus 1 GW-potential G via localization formulas can be compared to suitable genus 0 GW-invariants. As we shell see, the total contribution of all fixed point components with Σ_0 mapped to the fixed point α in X equals $\log \Delta_\alpha/48 - c_{-1}^\alpha u_\alpha/24$. The proof is partly based on intersection theory in Deligne-Mumford spaces.

Contributions of the second type fixed points to G are hard to compare with genus 0 GW-invariants directly because of the rotational symmetry of the cycle. By computing their contributions to the partial derivatives $\partial_\gamma G$ instead, we distinguish a marked point in (Σ, ε) which carries the class ϕ_γ and is situated on a branch of the graph approaching the cycle of 1-dimensional orbits at a fixed point α. This breaks the rotational symmetry of the cycle and allows us to compare localization formulas for the cycles with those for the chains between the fixed points α and β with $\beta = \alpha$. As we shell see, all such contributions add up to $R_{\alpha\alpha}^{(0)}\partial_\gamma u_\alpha/2$. The proof is based on "materialization" of Dubrovin's structural theory of semi-simple Frobenius manifolds in terms of fixed point localization.

Intersection theory in $\overline{\mathcal{M}}_{0,k}$ and $\overline{\mathcal{M}}_{1,k}$.

Let us consider the GW-theory with a point taken on the role of the target space (so that the moduli spaces of stable maps are Deligne-Mumford spaces) and introduce the following GW-potentials:

$$u(T) := \sum_{n=1}^{\infty} \frac{1}{n!}(1, T, ..., T, 1)_{n+2} ,$$

$$s(T, \hbar) := 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(1, T, ..., T, \frac{1}{\hbar - c})_{n+2} ,$$

$$v(T, x, y) := \frac{1}{x+y} + \sum_{n=1}^{\infty} \frac{1}{n!}(\frac{1}{x - c}, T, ..., T, \frac{1}{y - c})_{n+2} ,$$

$$\delta(T) := \sum_{n=0}^{\infty} \frac{1}{n!}(1, 1, 1, T, ..., T)_{n+3} ,$$

$$\mu(T) := \sum_{n=1}^{\infty} \frac{1}{n!}\omega[T, ..., T]_n ,$$

26

$$\nu(T) := \sum_{n=1}^{\infty} \frac{1}{n!}[T, ..., T]_n .$$

In these formulas, $T = t_0 + t_1 c + t_2 c^2 + ...$ is a series in one variable to be replaced by the 1-st Chern class $c^{(i)}$ of the universal cotangent line over $\overline{\mathcal{M}}_{g,n}$ with the index i depending on the position of the series T in the correlator. The correlators $(...)_k$ and $[...]_k$ mean integration over $\overline{\mathcal{M}}_{0,k}$ and $\overline{\mathcal{M}}_{1,k}$ respectively. The correlator $\omega[...]_k$ means integration

$$\int_{\overline{\mathcal{M}}_{1,k}} \omega \wedge ...$$

against the 1-st Chern class ω of the *Hodge line bundle* $\mathcal{H}$ over $\overline{\mathcal{M}}_{1,k}$. The fiber of this (orbi)-bundle over $[\Sigma, \varepsilon]$ is the space $H^0(\Sigma, K_\Sigma)$ of "holomorphic differentials" on the stable curve Σ. It is the pull-back of the Hodge line bundle over $\overline{\mathcal{M}}_{1,1}$ by (any of) the forgetting maps $\overline{\mathcal{M}}_{1,k} \to \overline{\mathcal{M}}_{1,1}$. Respectively, $\omega \wedge \omega = 0$, the class ω on $\overline{\mathcal{M}}_{1,1}$ coincides with $c^{(1)}$, and the orbi-structure of $\overline{\mathcal{M}}_{1,1}$ and $\mathcal{H}$ manifests in the well-known formula

$$\int_{[\overline{\mathcal{M}}_{1,1}]} \omega = 1/24 .$$

The potentials $u, s, v, \delta, \mu, \nu$ can be considered as functionals on the space of formal series T. We will assume however that the coefficients $t_0, t_1, t_2, ...$ are elements of some formal series algebra $K[[\Lambda]]$ (in our applications $K = \mathbb{Q}(\lambda)$), and that the whole series $T(c)$ can be rewritten as a formal q-series $\sum_{d \in \Lambda} a_d(c) q^d$ with coefficients a_d which are rational functions of c regular at $c = 0$. Thus each t_i is a formal q-series, and we will assume also, that $t_i \equiv 0$ mod (q) for $i > 0$. These conditions (satisfied in our applications) guarantee that the tragectory of the vector field

$$\mathcal{L} := \partial/\partial t_0 - t_1 \partial/\partial t_0 - t_2 \partial/\partial t_1 - ...$$

with the initial condition T is well-defined by

$$t_0(\tau) = \tau + \sum_{n=0}^{\infty} t_n(0) \frac{(-\tau)^n}{n!}, \ \ t_1(\tau) = 1 - dt_0/d\tau, \ \ t_2(\tau) = -dt_1/d\tau, ...$$

and has a unique intersection with the hyperplane $t_0 = 0$. We will use these facts in the following application of the string equation (notice that $\mathcal{L}T = 1 - (T(c) - T(0))/c$).

Proposition 3.1 (see [7, 12, 6]).

$$s = e^{u/\hbar}, \ v = \frac{e^{u/x+u/y}}{x+y}, \ \mu = \frac{u}{24}, \ \nu = \frac{\log \delta}{24} \ .$$

Proof. The string equation implies

$$\mathcal{L}u = 1, \ \mathcal{L}s = \frac{s}{\hbar}, \ \mathcal{L}v = \frac{v}{x} + \frac{v}{y},$$

$$\mathcal{L}\delta = 0, \ \mathcal{L}\mu = \frac{1}{24}, \ \mathcal{L}\nu = 0.$$

At $t_0 = 0$ the initial conditions

$$u = 0, \ s = 1, \ v = 1/(x+y), \ \delta = 1/(1-t_1), \mu = 0, \ \nu = -(\log(1-t_1))/24$$

can be computed from definitions with the use of dimensional reasoning ($\dim M_{0,n+2} = n - 1 < n$ and $\dim M_{1,n} = n$) and in the case of δ and ν — on the basis of the formulas $(1,1,1,c,...,c)_{n+3} = n!(1,1,1) = n!$ and $[c,...,c]_n = (n-1)![c] = (n-1)!/24$ which follow from the famous *dilation equation* $\langle T, ..., T, c \rangle_{g,n+1,d} = (2g - 2 + n)\langle T, ..., T \rangle_{g,n,d}$.

Materialization of canonical coordinates. We have to review here some basic structural results from [12] on localization in genus 0 equivariant GW-theory which were perhaps overshadowed by mirror theorems proved there. Let us outline here the key observation and refer to [12], Section 8, for further details.

Consider a stable map $f : (\Sigma, \varepsilon) \to X$ from a generic fiber of the contraction map $X_{0,n,d} \to \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,k}$ defined by the first k marked points. The fiber is specified by a configuration of k ordered distinct points on $\mathbb{C}P^1$. Due to the definition [19] of the contraction map, the curve Σ must contain a unique irreducible component $\mathbb{C}P^1$ carrying k special (= marked or singular) points realizing an isomorphic configuration. If the map f represents a fixed point of the torus action on $X_{0,n,d}$, then f maps the irreducible component to one of the fixed points in X. Thus we can single out the connected component

Σ_0 of $f^{-1}(X^G)$ (we will call it *special*), containing this $\mathbb{C}P^1$, and associate the fixed point $\alpha = f(\Sigma_0) \in X^G$ to the fixed point $[f] \in X^G_{0,n,d}$. This allows us to partition certain GW-invariants into contributions — via fixed point localization formulas — of those fixed points for which the special component is mapped to a given fixed point α. Applying the WDVV-argument to such contributions separately for each α we arrive at some *local* WDVV-identities which are essentially independent on the global WDVV-equation. Combining local and global WDVV-equations we obtain simultaneous diagonalization of quantum cup-product operators in a basis associated with fixed points of G in X.

Let us introduce the local GW-potentials $u_\alpha, D_\alpha, \Psi^\alpha_\beta$ involved into the diagonalizing structure. Let $t = \sum_\alpha t_\alpha \phi_\alpha$ denote the general equivariant cohomology class of X represented in the basis of fixed points. We denote ∂_α the partial derivative with respect to t_α and use the notation ∂_0 for the differentiation operator $\sum_\alpha \partial_\alpha$ in the direction of $1 \in H^*_G(X)$. We put $e_\alpha := \langle \phi_\alpha, \phi_\alpha \rangle^{-1} = \eta^{\alpha\alpha} = Euler_G(T_\alpha X)$.

- Consider a point in $X_{0,n,d}$ with the property that the first two marked points are located on the same connected component of $f^{-1}(X^G)$. The total contribution of all such fixed points to the GW-potential

$$e_\alpha \partial_\alpha \partial_\alpha F_0 = e_\alpha \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d\in\Lambda} q^d (\phi_\alpha, \phi_\alpha, t, ..., t)^d_{n+2}$$

  is denoted u_α. The potentials u_α have homogeneity degree 1, are congruent to t_α modulo (q) and can be taken on the role of local coordinates on $H^*_G(X)$ instead of t_α.

- Similarly, consider those fixed points where the first 3 marked points are located on the same connected component of $f^{-1}(X^G)$. The total contribution of such fixed points to the GW-potential

$$e_\alpha \partial_\alpha \partial_\alpha \partial_\alpha F_0 = e_\alpha \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d\in\Lambda} (\phi_\alpha, \phi_\alpha, \phi_\alpha, t, ..., t)^d_{n+3}$$

  is denoted D_α. We have: $\deg D_\alpha = 0$ and $D_\alpha \equiv 1 \mod (q)$.

- Finally, consider the fixed points with the 1-st marked point situated on the same connected component of $f^{-1}(X^G)$ as the two special points

which give birth to the branches carrying the 2-nd and 3-rd marked points. The total contribution of such fixed points to

$$e_\alpha \partial_\alpha \partial_\beta \partial_0 F_0 = e_\alpha \sum_{n=0}^\infty \frac{1}{n!} \sum_{d \in \Lambda} q^d (\phi_\alpha, \phi_\beta, 1, t, ..., t)_{n+3}^d$$

is denoted Ψ_β^α, has degree 0 and reduces to $\delta_{\beta\alpha}$ modulo (q).

Theorem 3.2. (see [12]). *The matrix* (Ψ_α^i) *satisfies the orthogonality relations*

$$\sum_i \Psi_\alpha^i e_i^{-1} \Psi_\beta^i = \delta_{\alpha\beta} e_\beta^{-1} \ , \ \sum_\alpha \Psi_\alpha^i e_\alpha \Psi_\alpha^j = \delta_{ij} e_j,$$

the normalization condition

$$\sum_\alpha \Psi_\alpha^i = D_i^{-1}$$

and diagonalizes structural constants of the quantum multiplication:

$$\partial_\alpha \partial_\beta \partial_\gamma F_0 = \sum_{i \in X^G} \Psi_\alpha^i \frac{D_i \Psi_\beta^i}{e_i} \Psi_\gamma^i.$$

The eigen-values $D_i \Psi_\beta^i$ *of the quantum cup-product operators* $\phi_\beta \circ$ *satisfy the integrability condition*

$$\sum_\beta D_i \Psi_\beta^i dt_\beta = du_i.$$

The theorem means that the local GW-potentials u_i are the canonical coordinates of Dubrovin's axiomatic theory of Frobenius structures, and D_i are the square roots of the normalized "Hessians": $\Delta_i = e_i D_i^2$.

Proof of Theorem 2.1. Let us begin with a remark on the general structure of fixed point localization formulas in $X_{g,n,d}$. A connected component of the fixed point set $X_{g,n,d}^G$ is identified by the combinatorial structure of a stable map $f : (\Sigma, \varepsilon) \to X$. The combinatorial structure can be specified by the following data:

- the genera g_i of connected components Σ_i of $f^{-1}(X^G)$ (we will call Σ_i *vertices* and consider the genus g_i undefined in the case if the vertex Σ_i is a point),

- the fixed points $\alpha = f(\Sigma_i)$,

- the graph of rational components of Σ connecting Σ_i's (we will call such components *edges*),

- the 1-dimensional orbits in X to which the edges are mapped to and the multiplicities of the maps (when necessary we will specify the orbit by the indices $\alpha \neq \beta$ of the fixed points it connects, denote $d_{\alpha\beta} \in \Lambda$ the degree of the orbit as a curve in X and denote m the multiplicity of the map),

- the indices of marked points situated on each Σ_i.

The connected components of $X^G_{g,n,d}$ are orbifolds (quotients of products of Deligne-Mumford spaces), and the virtual normal bundles whose equivariant Euler classes occur in the localization formulas are orbi-bundles over these orbifolds. These bundles can be split into virtual sums of contributions corresponding to the edges and to the vertices, and the Euler classes — into products of corresponding contributions. The contribution of each edge to the Euler class has the form of the product of characters of $LieG$ and depends only on the correspondind degree $d_{\alpha\beta}$ and multiplicity m.

Let us consider the intersection point x of a vertex Σ_i with an edge $\mathbb{C}P^1$. The virtual normal space contains the summand $T_x\Sigma_i \otimes T_x\mathbb{C}P^1$. It contributes to the *inverse* Euler class by $(\chi_{\alpha\beta}/m - c)^{-1}$. Here $\chi_{\alpha\beta}$ is the character of the torus action on the tangent line to the closure of the 1-dimensional orbit at $\alpha = f(\Sigma_i)$, and c is the 1-st Chern class of the universal cotangent line over $\overline{\mathcal{M}}_{g_i,k_i}$ at the marked point corresponding to x. The product of such contributions is to be integrated over $\overline{\mathcal{M}}_{g_i,k_i}$ in the localization formulas.

Adding up the contributions of all fixed point components in all the moduli spaces $X_{g,n,d}$ with various n and d to certain local GW-potentials we will obtain the exponential-like sums $\sum_k \langle T, ..., T \rangle / k!$ of integrals over the $\overline{\mathcal{M}}_{g,k}$ with rather complicated (and unspecified) series $T = t_0 + t_1 c + t_2 c^2 + ...$ For example, the local genus 0 GW-potential u_α equals $u(T)$. Here the q-series $T = \sum_d a_d(c)q^d$ has some rational functions of c on the role of the coefficients a_d and $a_0 = t_\alpha$. The whole series T takes in account contributions via localization formulas of all tree-like branches of genus 0 maps $f : (\Sigma, \varepsilon) \to X$ which join the special component Σ_0 at the fixed point α.

With the above remarks in mind, let us study now the contributions to the genus 1 GW-potential G of all the first type fixed points whose elliptic vertex Σ_0 is mapped to $\alpha \in X$. The contribution of the vertex to the virtual

normal bundle contains the summand

$$H^0(\Sigma_0, \mathcal{O}_{\Sigma_0} \otimes T_\alpha X) \ominus H^1(\Sigma_0, \mathcal{O}_{\Sigma_0} \otimes T_\alpha X).$$

Thus the inverse Euler class contains the factor

$$\frac{Euler_G(\mathcal{H}^* \otimes T_\alpha X)}{Euler_G(T_\alpha X)} = 1 - c^\alpha_{-1}\omega$$

in addition to the factors $(\chi_{\alpha\beta}/m - c)^{-1}$ discussed above. The total contribution of the first type fixed points equals therefore

$$\sum [T, ..., T]/k! - c^\alpha_{-1} \sum \omega[T, ..., T]/k! = \nu(T) - c^\alpha_{-1}\mu(T) \ .$$

Notice that the series T here is the same as in the above description of u_α. We conclude that the total contribution equals

$$(\log D_\alpha)/24 - c^\alpha_{-1}u_\alpha/24 \ .$$

Consider now the contributions of the second type fixed points to

$$\partial_\gamma G = \sum_{n,d} q^d[\phi_\gamma, t, ..., t]^d_{n+1}/n! \ .$$

Let α be the fixed point in X where the tree-like branch of (Σ, ε) carrying the 1-st marked point (with the class ϕ_γ) joins the cycle of edges in Σ, and let Σ_0 be the corresponding vertex of Σ. Denote χ and χ' the characters of $LieG$ on the tangent lines to the 1-dimensional orbits where the edges of the cycle adjecent to Σ_0 are mapped to, and denote m and m' the corresponding multiplicities. Summing over all the second type fixed points with these data we see that the contribution of the vertex Σ_0 can be described as $e_\alpha^{-1}\partial_\gamma v(T, \chi/m, \chi'/m')$

$$= \partial_\gamma \frac{\exp(u_\alpha m/\chi + u_\alpha m'\chi')}{(\chi/m + \chi'/m')e_\alpha} = \frac{\exp(u_\alpha m/\chi + u_\alpha m'/\chi')}{(\chi/m)(\chi'/m')e_\alpha} \, \partial_\gamma u_\alpha.$$

This localization factor can be rewritten as

$$(\partial_\gamma u_\alpha) \lim_{x,y \to 0} \frac{\exp(u_\alpha m/\chi)}{e_\alpha(x + \chi/m)} \, e_\alpha \, \frac{\exp(u_\alpha m'/\chi')}{e_\alpha(y + \chi'/m')} \ .$$

Let us compare now this localization factor and the contribution of the rest of the cycle with localization formulas for the genus 0 GW-potential

$$V_{\alpha\alpha}(x, y) = \sum_{n,d} \frac{q^d}{n!}(\frac{\phi_\alpha}{x - c}, t, ..., t, \frac{\phi_\alpha}{y - c})^d_{n+2} \ .$$

The localization factors corresponding to vertices carrying the first and the last marked points (with the classes ϕ_α) vanish unless these vertices are mapped to $\alpha \in X$. If they are, consider the chain of edges connecting the vertices and denote m, m' and χ, χ' the multiplicities and the characters of the edges adjecent to these vertices. The localization factors of the vertices with these data are equal to

$$e_\alpha^{-1}v(T, x, \chi/m) = \frac{\exp(u_\alpha/x + u_\alpha m/\chi)}{e_\alpha(x + \chi/m)},$$

$$e_\alpha^{-1}v(T, y, \chi'/m') = \frac{\exp(u_\alpha/y + u_\alpha m'/\chi')}{e_\alpha(y + \chi'/m')}.$$

Since the rest of the chain contributes to $\partial_\gamma G$ and to $V_{\alpha\alpha}$ in the same way, we conclude that the total contribution to $\partial_\gamma G$ of the second type fixed point in question equals

$$(\partial_c u_\alpha)\frac{1}{2} \lim_{x,y\to 0} [e^{-u_\alpha/x}V_{\alpha\alpha}(x, y)e^{-u_\alpha/y}e_\alpha - \frac{1}{x + y}] \ .$$

where the factor $1/2$ takes care of the two orientations of cycles.

Let us look now at the fundamental solution matrix

$$S_{\beta\alpha} = \sum_{n,d} \frac{q^d}{n!}(\phi_\beta, t, ..., t, \frac{\phi_\alpha}{\hbar - c})^d_{n+2}$$

via localization formulas. The dependence of $S_{\beta\alpha}$ on $\hbar$ is due only to the localization factor of the vertex carrying the last marked point; it is equal to

$$\frac{\exp(u_\alpha/\hbar)}{e_\alpha(\hbar + \chi/m)}$$

if the first marked point belongs to another vertex, and to $\exp(u_\alpha/\hbar)e_\alpha^{-1}\delta_{\beta\alpha}$ if the vertex is the same. Since $\chi \neq 0$, we can expand $(\hbar + \chi/m)^{-1}$ into a

power series in $\hbar$ and summing over all fixed point conponents obtain the asymptotical expansion $\Psi(1 + \hbar R^{(0)} + o(\hbar)) \exp(U/\hbar)$ of Proposition 1.1 for the fundamental solution matrix $(S_{\beta\alpha}e_\alpha)$. Notice that the matrix Ψ of eigenvectors here is normalized in the same way as (and thus coincides with) the matrix (Ψ^α_β) in Theorem 3.2, since $S_{\beta\alpha}e_\alpha \equiv \delta_{\alpha\beta} \mod (q)$.

It remains only to invoke the WDVV-identity

$$V_{\alpha\alpha} = \sum_\beta S_{\beta\alpha}(x)e_\beta S_{\beta\alpha}(y)/(x+y) \; ,$$

the asymptotical expansion

$$S_{\beta\alpha}(\hbar) = \sum_i \Psi^i_\beta (\delta_{i\alpha} + \hbar R^{(0)}_{i\alpha} + o(\hbar)) \; e^{u_\alpha/\hbar} e^{-1}_\alpha \; ,$$

and the orthogonality relation

$$\sum_\beta \Psi^i_\beta e_\beta \Psi^j_\beta = \delta_{ij} e_i$$

in order to identify the above limit with $R^{(0)}_{\alpha\alpha}$.

Combining the contributions of all first and second type fixed points we conclude that

$$dG = \sum_\alpha [\; d(\log D_\alpha)/24 - c^\alpha_{-1} du_\alpha/24 + R^{(0)}_{\alpha\alpha} du_\alpha/2 \;] \; .$$

4   A mirror theory for concave bundles.

Genus 1. Let X be a compact Kähler manifold and V be a holomorphic vector bundle $E \to X$ with the total space E. We call the bundle V *concave* if for any non-constant stable map $f : (\Sigma, \varepsilon) \to X$ the induced bundle f^*V over Σ has no global holomorphic sections: $H^0(\Sigma, f^*V) = 0$. Direct sums of negative line bundles are concave and will play the role of main examples in this section.

If V is concave then non-constant stable maps to E are actually maps to the zero section of V and therefore the moduli spaces $E_{g,n,d} = X_{g,n,d}$ are compact for $d \neq 0$. This allows one to define GW-invariants of non-compact

space E. Namely, for $d \neq 0$ denote $V'_{g,n,d}$ the *obstruction* bundle over $X_{g,n,d}$ formed by the spaces $H^1(\Sigma, f^*V)$. The virtual fundamental class $[E_{g,n,d}]$ is the cap-product of $[X_{g,n,d}]$ with the Euler class of the obstruction bundle:

$$\int_{[E_{g,n,d}]} \Phi = \int_{[X_{g,n,d}]} \Phi \wedge Euler(V'_{g,n,d}).$$

In order to include the concave bundle spaces into the general framework of GW-theory one has to extend the above formula to the case $d = 0$ when the moduli spaces are non-compact. Following [12] we provide V with the fiberwise circle action $U_1 : E$ by unitary scalar multiplication. The constant maps $f : (\Sigma, \varepsilon) \to X$ form the fixed point set $X \times \overline{\mathcal{M}}_{g,n}$ of U_1-action on $E_{g,n,0} = E \times \overline{\mathcal{M}}_{g,n}$ with the normal bundle $V \otimes \mathbb{C} = H^0(\Sigma, f^*V)$. We introduce U_1-equivariant GW-invariants of E for $d \neq 0$ — by

$$\int_{[E_{g,n,d}]} \Phi = \int_{[X_{g,n,d}]} \Phi \wedge Euler_{U_1}(V'_{g,n,d}),$$

and for $d = 0$ — by the localization formula

$$\int_{[E_{g,n,d}]} \Phi := \int_{[X_{g,n,d}]} \Phi \wedge \frac{Euler_{U_1}(V'_{g,n,d})}{Euler_{U_1}(V)}.$$

The GW-invariants take values in the coefficient field $\mathbb{Q}(\lambda)$ of the U_1-equivariant theory, but the degree $d \neq 0$ invariants are defined over $\mathbb{Q}[\lambda]$ and specialize to the non-equivariant ones at $\lambda = 0$. The construction immediately extends to the case of GW-theory equivariant with respect to an additional group G acting on $E \to X$.

As it is shown in [12] [15] the genus 0 GW-invariants of concave bundle spaces E have the same properties as equivariant GW-invariants of compact manifolds including WDVV, string and divisor equations. In particular they define on the space $H = H^*(E, \mathbb{Q}(\lambda)[[\Lambda]])$ provided with the intersection pairing

$$\langle \phi, \psi \rangle = \int_X \phi \wedge \psi \wedge Euler_{U_1}^{-1}(V)$$

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[15]Strictly speaking the paper deals with the case of convex base X, but the arguments easily extend to any Kähler base as soon as the GW-theory for X has been worked out.

a quasi-conformal Frobenius structure of dimension $D = \dim_{\mathbb{C}} E$. The Euler vector field on H is defined by the usual rules; in particular $\deg \lambda = 1$, and the 1-st Chern class of the tangent bundle is $c_1(\mathcal{T}_E) = c_1(\mathcal{T}_X) + c_1(V)$.

We generalize our genus 1 theory to concave bundle spaces E. Similarly to the compact case we introduce potential

$$G(t, q, \lambda) := \sum_{n,d} \frac{q^d}{n!} [t, ..., t]_d^n$$

encoding genus 1 equivariant GW-invariants of E. The Conjecture 1.2 applies.

Let us assume now that the base X is provided with a Killing Hamiltonian action of a torus T, that the action of the complexified torus has only isolated 0- and 1-dimensional orbits, and that the action can be lifted to the bundle $E \to X$. The Frobenius structure defined on H by the genus 0 GW-invariants of E is genericly semi-simple. In particular, the canonical coordinates u_α, the "Hessians" Δ_α and the asymptotical coefficients $R_{\alpha\alpha}^{(0)}$ are defined.

The reduction of the genus 1 potential G modulo (q) equals $-\sum_\alpha c_{-1}^\alpha t_\alpha / 24$ where $\sum t_\alpha \phi_\alpha$ is the coordinate representation of the general cohomology class t in the basis of δ-functions of the fixed points, and c_{-1}^α is the ratio $c_{\dim E - 1}(T_\alpha E) / c_{\dim E}(T_\alpha E)$ of the $T \times U_1$-equivariant Chern classes of the tangent space to E at the fixed point α.

The following theorem — and its proof — is a straightforward generalization of Theorem 2.1 to concave bundle spaces.

Theorem 4.1.

$$dG = \sum_\alpha [\, d(\log \Delta_\alpha)/48 - c_{-1}^\alpha du_\alpha/24 + R_{\alpha\alpha}^{(0)} du_\alpha/2].$$

Genus 0. We develop now a mirror theory of concave *toric* bundle spaces which in principal allows one to compute their genus 0 GW-invariants.

According to T. Delzant, a compact symplectic toric manifold X with the Picard number r can be obtained by the symplectic reduction of a standard $\mathbb{C}^m$ by a linear torus action $T^r : \mathbb{C}^m$ on a sutable level of the momentum map. [16] According to F. Kirwan, the equivariant cohomology algebra $H_{T^m}^*(X)$ with

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[16] We refer to [14] for a detailed discussion of combinatorics, geometry and topology of symplectic toric manifolds.

respect to the maximal torus action $T^m : \mathbb{C}^m$ is generated over $\mathbb{Q}[\lambda_1, ..., \lambda_m] = H^*(BT^m)$ by the classes $w_1, ..., w_m$ of degree 2 Poincaré-dual to the invariant cycles obtained by the reduction of the coordinate hyperplanes in $\mathbb{C}^m$. The generators w_j can be written as linear combinations

$$w_j = \sum_{i=1}^{r} p_i m_{ij} - \lambda_j, \ j = 1, ..., m,$$

in terms of some classes $p_1, ..., p_r$ representing a basis in $H^2(X, \mathbb{Z})$. We will use this basis for labeling by $d = (d_1, ..., d_r)$ the degrees of curves in X and denote Λ the semigroup of the degrees.

Consider the concave vector bundle $V : E \to X$ which is the direct sum of l *negative* line bundles over X. Let

$$v_j = \lambda'_j - \sum_{i=1}^{r} p_i l_{ij}, \ j = 1, ..., l,$$

be the 1-st Chern classes of the summands equivariant with respect to the torus $G := T^m \times T^l$ action where the second factor acts fiberwise on V by diagonal transformations.

Our objective is to compute the fundamental solution matrix $(S_{\alpha\beta}(t, q, \hbar))$ for the concave bundle space E at $q = 1$ and $t \in H^0_G(E) \oplus H^2_G(E)$. According to the string and divisor equations it coincides with

$$S_{\alpha\beta} := \sum_{d \in \Lambda} q^d (\phi_\alpha, e^{(t_0 + p \log q)/\hbar} \frac{\phi_\beta}{\hbar - c})_d,$$

where $\{\phi_\alpha\}$ is the basis in $H^*_G(E)$ of δ-functions of the fixed points, $q^d = \exp(\sum d_i t_i)$, $p \log q = \sum p_i t_i$, $t_1, ..., t_r$ are coordinates on $H^2(E)$ and t_0 is the coordinate on $H^0(E)$.

We introduce the formal series in q and $1/\hbar$ with vector coefficients in $H^*_G(E, \mathbb{Q}(\lambda, \lambda'))$ by

$$J := 1 + \frac{1}{\hbar} \sum_{d \neq 0} q^d \, \mathrm{ev}_* \frac{Euler_G(V'_{0,1,d})}{\hbar - c}$$

where ev : $E_{0,1,d} \to E$ is the evaluation map. The vector-function J is related to the row sum of the matrix $(S_{\alpha\beta})$ by

$$\sum_\alpha S_{\alpha\beta} = \langle J, e^{(t_0 + p\log q)/\hbar}\phi_\beta\rangle.$$

This follows from the string equation in view of $\sum \phi_\alpha = 1$.

We introduce the hypergeometric series I in q and $1/\hbar$ with coefficients in $H_G^*(X, \mathbb{Q}(\lambda, \lambda'))$ by the following explicit formula:

$$I := \sum_{d\in\Lambda} q^d \Pi_{j=1}^l \frac{\Pi_{k=-\infty}^{L_j(d)-1}(v_j - k\hbar)}{\Pi_{k=-\infty}^{-1}(v_j - k\hbar)} \, \Pi_{j=1}^m \frac{\Pi_{k=-\infty}^0(w_j + k\hbar)}{\Pi_{k=-\infty}^{D_j(d)}(w_j + k\hbar)}$$

where $L_j(d) = \sum_i d_i l_{ij}$, $D_j(d) = \sum_i d_i m_{ij}$.

Theorem 4.2. *Suppose that the 1-st Chern class $\sum_{j=1}^m w_j + \sum_{j=1}^l v_j$ of the concave toric bundle space E is non-negative. Then*

$$e^{(t_0 + p\log q)/\hbar}J \quad and \quad e^{(t_0 + p\log q)/\hbar}I$$

coincide up to a change of variables

$$t_0 \mapsto t_0 + f_0(q) + \sum \lambda_j g_j(q) + \sum \lambda'_j h_j(q),$$

$$\log q_i \mapsto \log q_i + f_i(q), \ \ i = 1, ..., r,$$

where f_i, g_j, h_j (resp. f_0) are q-series supported at $\Lambda - 0$ of the homogeneity degree 0 (resp. 1).

Remark. The hypothesis $c_1(\mathcal{T}_E) \geq 0$ guarantees that $\deg q^d \geq 0$ for any $d \in \Lambda$. The series I and J have the homogeneity degree 0. By definition $J = 1 + o(\hbar^{-1})$. The change of variables transforming I to J is determined by the asymptotics

$$I = 1 + \hbar^{-1}[f_0 + \sum p_i f_i + \sum \lambda_j g_j + \sum \lambda'_j h_j] + o(\hbar^{-1}).$$

Theorem 4.2 is quite similar to the mirror theorem 0.2 in [14] for toric convex super-manifolds and complete intersections. However in the case of concave bundles V of dimension $l > 1$ the series I has the asymptotics $1 + o(\hbar^{-1})$ (due to the factor $\Pi_{j=1}^l(v_j - 0\hbar)$) and thus coincides with J.

Corollary 4.3. *For concave toric vector bundles $E \to X$ of dimension > 1 the GW-potential J of the total space E coincides with the hypergeometric series I, provided that $c_1(\mathcal{T}_E) \geq 0$.*

Actually $I = 1 + O(\hbar^{-l})$ which allows us to derive

Corollary 4.4. *If $l > 1$ then in the small equivariant quantum cohomology algebra of E we have $v_1 \circ ... \circ v_l = \varphi(\pm q) v_1 ... v_l$ where $\pm q^d = (-1)^{\sum L_j(d)} q^d$ and*

$$\varphi(q) = \sum_{d: \sum L_j(d) = \sum D_j(d)} \frac{L_1(d)!...L_l(d)!}{D_1(d)!...D_m(d)!} q^d,$$

while any shorter quantum product of degree 2 classes coincides with classical.

Proof. We are going to exploit the fact, that the matrix $S = (S_{\alpha\beta} e_\beta^{-1})$ satisfies $\hbar \partial S = (p \circ) S$, where ∂ is the derivative $q \partial / \partial q$ in the direction of a second degree class p, as the base for induction. Suppose that for all p we have $(\hbar \partial)^k S = (p^k \circ) S$ and thus $= (p^k \circ) + O(\hbar^{-1})$ with no terms of positive order in $\hbar$. Then the row sums $\sum_\alpha \langle \phi_\alpha, p^k \circ \phi_\beta \rangle = \langle p^k, \phi_\beta \rangle$ are constant, and we conclude that

$$(\hbar \partial)^{k+1} S = \hbar(\partial(p^k \circ))S + (p^k \circ p) \circ S$$

has the row sums $\langle p^k \circ p, \phi_\beta \rangle + O(\hbar^{-1})$. Therefore, if the row sum of S has the asymptotics $(1 + O(\hbar^{-l})) e^{(t_0 + p \log q)/\hbar}$ (as in our case) we conclude by induction that for $k < l - 1$ we have $\langle p^k \circ p, \phi_\beta \rangle = \langle p^{k+1}, \phi_\beta \rangle$ for any β and thus $p^k \circ p = p^{k+1}$. In the border case $k = l - 1$ we find $p^k \circ p$ from the row sum of $(\hbar \partial)^{k+1}$ modulo $\hbar^{-1}$.

In the case of the row sum $I e^{(t_0 + p \log q)/\hbar}$ we apply polarization of the above conclusion and consecutively differentiate in the directions corresponding to the classes $p = v_1, ..., v_l$. The resulting series equals $v_1 ... v_l \varphi(\pm q)$ modulo $\hbar^{-1}$, and thus $v_1 \circ ... \circ v_l = v_1 ... v_l \varphi(\pm q)$. $\square$

While the present paper was in preparation, a result equivalent to Theorem 4.2 in the case of concave bundles over projective spaces was published in [22]. [17]

We outline below a proof of Theorem 4.2 which is completely parallel to the proof of the mirror theorem for projective and toric complete intersections

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[17]In our lecture course at UC Berkeley [28] the theorem was also stated over projective spaces.

given in [12] and [14] respectively and, as we explain in the footnotes, is a variant of the proof given in [22].

Scheme of the proof. *Step* 1. Fixed point localization in $E_{0,2,d}$ gives rise to a recursion relation for $S_{\alpha\beta}$. Namely, introduce the formal series J_α^β in q and $1/\hbar$ with coefficients in $\mathbb{Q}(\lambda,\lambda')$ by

$$J_\alpha^\beta(q,\hbar) := S_{\alpha\beta}(q,\hbar)e^{-(t_0+p(\beta)\log q)/\hbar}e_\beta,$$

where $p(\beta)$ is the localization of p at the fixed point $\beta \in E$, and $e_\beta = Euler_G(T_\beta E)$. As a $1/\hbar$-series, $J_\alpha^\beta = \delta_{\alpha\beta}+O(\hbar^{-1})$ by definition, but $\sum_\alpha J_\alpha^\beta = 1 + o(\hbar^{-1})$ since the row sum of the matrix $J_{\alpha\beta}$ is equal to the localization J^β of the vector-function J at the fixed point $\beta \in E$.

Proposition 4.5.

$$J_\alpha^\beta(q,\hbar) = \delta_{\alpha\beta} + \sum_{d\neq0} q^d P_\alpha^{\beta\ (d)}(\hbar^{-1}) + \sum_{\gamma\neq\beta}\sum_{m=1}^\infty J_\alpha^\gamma(q,\chi_{\gamma\beta}/m)\frac{q^{md_{\gamma\beta}}\ Coeff_\gamma^\beta(m)}{\hbar - \chi_{\gamma\beta}/m}$$

where $P_\alpha^{\beta\ (d)}$ are polynomials in $\hbar^{-1}$ with coefficients in $\mathbb{Q}(\lambda,\lambda')$ and $Coeff_\gamma^\beta(m)$ are (known) rational functions of (λ,λ').

The proof of Proposition 4.5 [12, 14] is obtained by counting contributions to J_α^β of fixed points in $E_{0,2,d}$ via localization formulas. The tree representing such a fixed point contains the vertices Σ_α and Σ_β carrying the two marked points and contains a chain of edges connecting these vertices. In the case when Σ_β is a point, the last edge in the chain (it connects the fixed points γ and β) yields the localization factor $Coeff_\gamma^\beta(m)/(\hbar - \chi_{\gamma\beta})$. The characteristics $m, \chi_{\gamma\beta}, d_{\gamma\beta}$ of the edge here are the same as in Section 3. The rest of the chain is taken care of by the localization factor $J_\alpha^\gamma(q,\chi_{\gamma\beta}/m)$. All other fixed points (with Σ_β being a curve) contribute somehow to the polynomial tail $\sum P^{(d)}q^d$.

Proposition 4.5 means that the coefficients of the q-series J_α^β are rational functions in $\hbar$ with 1-st order pole at $\hbar = \chi_{\gamma\beta}/m$ and the residue at the pole, controlled recursively by the (known) coefficients $Coeff_\gamma^\beta(m)$, and with high order pole at $\hbar = 0$.

The same recursion relation (with the index α omitted) holds true for the row sums J^β. There is no need here to write down explicitly the recursion coefficients $Coeff_\gamma^\beta(m)$. In fact the localization components I^β of the explicitly written vector-function I are also q-series with coefficients rational in $\hbar$.

Rewriting I^β as sums of simple fractions $1/(\hbar - \chi_{\gamma\beta}/m)$ yields a recursion relation for I^β of the same form as the one for J^β. It suffices to tell only that the recursion coefficients $Coeff_\gamma^\beta(m)$ for I^β are *the same* as for J^β (while the polynomial tails can be different).

Step 2. Given the polynomials $P_{\alpha\beta}^{(d)}(\hbar^{-1})$, the recursion relation of Proposition 4.5 determines the matrix (J_α^β) unambiguously. The following proposition provides a serious constraint on these polynomials.

Proposition 4.6. *For any α, γ the series in q and $z = (z_1, ..., z_r)$*

$$\sum_\beta J_{\alpha\beta}(qe^{\hbar z}, \hbar)e^{p_\beta z}e_\beta^{-1}J_{\beta\gamma}(q, -\hbar)$$

has coefficients polynomial in $\hbar$.

The proof of Proposition 4.6 (see [12, 14]) is based on another interpretation of GW-potentials $S_{\alpha\beta}$. Let us consider the concave bundle $\mathcal{E} \to \mathcal{X}$ which is the cartesian product of $E \to X$ with $\mathbb{C}P^1$ and is provided with the standard action of S^1 via the second factor. The $G \times S^1$-equivariant cohomology of $\mathcal{E}$ is isomorphic to the tensor product of S^1-equivariant cohomology algebra $\mathbb{Q}[\pi, \hbar]/(\pi(\pi - \hbar))$ of $\mathbb{C}P^1$ (here $\hbar$ is the generator of $H^*(BS^1)$) with the G-equivariant cohomology algebra of E. It has a basis $\{\phi_\alpha \pi, \phi_\gamma(\hbar - \pi)\}$.

Let $\mathcal{E}_{n,d}$ be the moduli space of genus 0 stable maps to $\mathcal{E} = E \times \mathbb{C}P^1$ with n marked points of degre d in projection to E and of degree 1 in projection to $\mathbb{C}P^1$. We introduce the GW-potential

$$\mathcal{G}_{\alpha\gamma} := \sum_{n,d} \frac{q^d}{n!}\langle \phi_\alpha \pi, pz\pi, ..., pz\pi, \phi_\gamma(\hbar - \pi)\rangle_d^{n+2},$$

where the correlator $\langle ...\rangle_d^{n+2}$ refers to the $G \times S^1$-equivariant GW-invariant of the concave bundle space $\mathcal{E}$ obtained by integration over $[\mathcal{E}_{n+2,d}]$.

The series $\mathcal{G}_{\alpha\gamma}$ depends on $\hbar$ but not on $\hbar^{-1}$ since it is defined without localization to fixed points of S^1-action. Applying localization to fixed points of S^1-action on $\mathcal{E}_{n,d}$ and then using the divisor equation one finds that $\mathcal{G}_{\alpha\gamma}$ coincides with the convolution series introduced in Proposition 4.6.

Remark. One can also define the GW-potentials $\mathcal{G}_{\alpha\gamma}$ by

$$\mathcal{G}_{\alpha\gamma} = \sum_d q^d \int_{[\mathcal{E}_{2,d}]} \mathrm{ev}_1^*(\phi_\alpha \pi)\, \mathrm{ev}_2^*(\phi_\gamma(\hbar - \pi))Euler_{G \times S^1}(\mathcal{V}_{2,d}')e^{Pz},$$

where $P = (P_1, ..., P_r)$ are the equivariant 1-st Chern classes of the *universal* line bundles over $\mathcal{E}_{2,d}$ introduced in [14]. The definition uses embeddings of the toric manifold into projective spaces and the map φ from $\mathcal{X}_{0,d}$ to the *toric compactification* $\mathcal{X}_{(d)}$ of spaces of degree d maps $\mathbb{C}P^1 \to X$ in the case when X is a projective space.

Summing $\mathcal{G}_{\alpha\gamma}$ over α and γ we arrive to the polynomiality property for the row sums J^β. The same polynomiality property holds true for I^β: $\langle I(qe^{\hbar z}, \hbar), e^{pz} I(q, -\hbar) \rangle$ depends on $\hbar$ but not on $\hbar^{-1}$. The proof (see [14]) is based on localization to fixed point of S^1-action applied to the series

$$\tilde{\mathcal{G}} = \sum_{[\mathcal{X}_{(d)}]} Euler_{G \times S^1}(\mathcal{V}'_{(d)}) e^{Pz}$$

which mimics the GW-potential $\sum_{\alpha\gamma} \mathcal{G}_{\alpha\gamma}$ in terms of toric compactifications $\mathcal{X}_{(d)}$ of spaces of degree d maps $\mathbb{C}P^1 \to X$.

Step 3. Let us call a solution (J^β_α) (respectively (J^β)) to the recursion relation of Proposition 4.5 *polynomial* if it satisfies the polynomiality property described in Proposition 4.6.

Proposition 4.7. *A polynomial solution (J^β_α) (respectively (J^β)) to the recursion relation satisfying the asymptotical conditions*

$$J^\beta_\alpha = \delta_{\alpha\beta} + O(\hbar^{-1}), \quad \sum_\alpha J^\beta_\alpha = 1 + o(\hbar^{-1})$$

(respectively $J^\beta = 1 + o(\hbar^{-1})$) is unique (if it exisits).

The proof is obtained by a straightforward argument of perturbation theory as in Proposition 4.5 in [14]. This result completes the proof of Corollary 4.3.

Remark. It is not hard to prove that under the hypotheses of Corollary 4.3 there exist differential operators $\mathcal{D}_\alpha(\hbar q \partial / \partial q, q, \hbar)$ such that

$$J^\beta_\alpha e^{(p_\beta \log q)/\hbar} = \mathcal{D}_\alpha[J^\beta e^{(p_\beta \log q)/\hbar}].$$

The proof is constructive, but we do not know how to describe the formulas for J^β_α in a closed form.

Step 4. Since both (J^β) and (I^β) are polynomial solutions to the same recursion relation, the proof of Theorem 4.2 is completed by the following proposition whose proof is also straightforward.

Proposition 4.8. *Transformations described in Theorem* 4.2 *preserve the class of polynomial solutions to the recursion relation.*

Remark. One can easily point out constant coefficient differential operators $\mathcal{D}_\alpha$ such that the hypergeometric series

$$I_\alpha^\beta := e^{(-p\log q)/\hbar}\mathcal{D}_\alpha[I^\beta e^{(p\log q)/\hbar}]$$

form a polynomial solution to the recursion relation of Proposition 4.5. They usually have wrong asymptotics however. The matrix (I_α^β) can be transformed to J_α^β, but in general the transformation requires matrix differential operators of infinite order (including changes of variables), and we do not know how to describe the transformation concisely. [18]

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[18]The genus 0 mirror conjecture for complete intersections in the projective space $X = \mathbb{C}P^n$ has now five proofs — the four variations of the same proof (in [12], in [14], the one outlined above but applied to convex bundles over X instead of concave bundles, and the one in Section 5 of this paper based on nonlinear Serre duality), and the proof recently given in [22]. Here we compare the methods in [22] with our approach.

The key idea (see Step 2 above) — to study GW-invariants of the product $X \times \mathbb{C}P^1$ *equivariant with respect to the S^1-action on $\mathbb{C}P^1$* instead of GW-invariants on X — is borrowed in [22] from our paper [12], Sections 6 and 11. In fact this idea is profoundly rooted in the heuristic interpretation [15] of GW-invariants of X in terms of Floer cohomology theory on the loop space LX where the S^1-action is given by rotation of loops. The generator in the cohomology algebra of BS^1 denoted $\hbar$ in our papers corresponds to α in [22].

Another idea, which is used in all known proofs and is due to M. Kontsevich [19], is to replace the virtual fundamental cycles of spaces of curves in a complete intersection by the Euler cycles of suitable vector bundles over spaces of curves in the ambient space. Both papers [12] and [22] are based on computing the push forward of such cycles to simpler spaces. Namely, the cycles are S^1-equivariant Euler classes of suitable bundles over stable map compactifications of spaces of bi-degree $(d, 1)$ rational curves in $X \times \mathbb{C}P^1$, the simpler spaces are *toric compactifications* of spaces of degree d maps $\mathbb{C}P^1 \to X = \mathbb{C}P^n$, and the push-forwards are denoted E_d in [12] and $\varphi_!(\chi_d)$ in [22].

The toric compactification is just the projective space $\mathbb{C}P^{(n+1)d+n}$ of $(n+1)$-tuples of degree $\leq d$ polynomials in one variable z, which generically describe degree d maps $\mathbb{C}P^1 \to X = \mathbb{C}P^n$; the space is provided with the S^1-action $z \mapsto z\exp(it)$ (as in the loop space!) Thus both papers depend on continuity of certain natural map (denoted μ in [12] and φ in [22]) between the two compactifications. The continuity is stated in [22] as Lemma 2.6. It coincides with our Main Lemma in [12], Section 11. Our proof of the lemma based on an inductive computation with line bundles over curves in terms of local trivializations is replaced in [22] by more standard algebraic-geometrical arguments due to J. Li. It is worth repeating here the remark from [12] that a different proof of the lemma

was provided to me by M. Kontsevich, with whom we first discussed the map between the two compactifications in Fall 1994.

The new concept introduced in [22] — the *eulerity* property of the classes E_d (Definition 2.3 in [22]) — is to replace both the *recursion* relation (Step 1 above) and the *polynomiality* property (Step 2) of the gravitational GW-invariant (J in the above outline). Eulerity is actually equivalent to recursion + polynomiality. Theorem 2.5 in [22] asserting the eulerity property of the classes $\{E_d\}$ coincides with Proposition 11.4(2) in [12] deduced there *from* the recursion + polynomiality. The proof of Theorem 2.5 in [22] is based on the same localization to fixed points of S^1-action on spaces of curves as in our proof of Corollary 6.2 in [12] which guarantees the polynomiality. The recursion is derived in [12] by further fixed point localization with respect to the torus acting on $X = \mathbb{C}P^n$. Thus the proof in [22] shows that the latter localization argument is unnecessary.

The relationship among the two solutions to the recursion relation — the gravitational GW-invariant and the explicitly defined hypergeometric series (I in the above outline) — is based on some uniqueness result (Proposition 11.5 in [12]) for solutions to the recursion relation satisfying the polynomiality property. The corresponding result in [22] is Theorem 2.11 about linked Euler data. *Linked* there translates to our terminology as the recursion coefficients in the recursion relations for I and J being the same. The proof of the uniqueness result in [22] is the same as in [12] or [14]. The difference is that the uniqueness property is formulated in [22] solely in terms of the Euler data $\{E_d\}$ and not in terms of gravitational GW-invariant the data generate.

The uniqueness result allows to identify the gravitational and hypergeometric solutions to the recursion by some changes of variables (the mirror transformations). This is deduced in [12] from Proposition 11.6 which states that both the recursion relation and the polynomiality property are preserved by the mirror transformation (see Step 4 above). The corresponding result in [22] is Lemma 2.15 which says that the (equivalent!) eulerity property is invariant under mirror transformations. It turns out however that while it is straightforward to check the invariance of recursion and polynomiality (Proposition 11.6 in [12]), it is technically harder to give a direct proof of the invariance of eulerity, which requires the notion of *lagrangian lifts* introduced in [22]. The use of lagrangian lifts is therefore unnecessary.

The last part of the proof in [22] (see Section 3 there) addresses the following issue: while the previous results allow to compute *some* GW-invariants in terms of hypergeometric functions, what do these GW-invariants have to do with the structural constants of quantum cohomology algebra involved in the formulation of the mirror conjecture?

The computational approach to the issue in [22] is also not free of overlaps with [12]. However it remains unclear *to us* why the authors of [22] ignore the fundamental relationship between the gravitational GW-invariant and quantum cohomology which resolves the issue momentarily. The relationship was described by R. Dijkgraaf and B. Dubrovin [7] in the axiomatic context of 2-dimensional field theories and adjusted to the setting of equivariant GW-theory in Section 6 of [12]. According to these results the structural constants of quantum cohomology algebra (such as *Yukawa coupling* in the case of quintic 3-folds) are *coefficients* of the linear differential equations satisfied by *the* gravitational

44

Mirrors. The hypergeometric series I^β can be represented by hypergeometric integrals:

$$I^\beta(q,\hbar)e^{(p_\beta \log q)/\hbar} = \int_{\Gamma_q^\beta \subset E_q'} e^{(\sum_{j=1}^m W_j + \sum_{j=1}^l V_j)/\hbar} \Pi_{j=1}^m W_j^{\lambda_j/\hbar} \Pi_{j=1}^l V_j^{-\lambda_j'/\hbar} \times$$

$$\times \frac{d\log W_1 \wedge ... \wedge d\log W_m \wedge d\log V_1 \wedge ... \wedge d\log V_l}{d\log q_1 \wedge ... \wedge d\log q_r}.$$

Here Γ_q^β are suitable non-compact cycles of middle dimension in the complex $m + l - r$-dimensional manifold

$$E_q' = \{(W,V)|\Pi_{j=1}^m W_j^{m_{ij}} = q_i \Pi_{j=1}^l V_j^{l_{ij}}, \ i = 1, ..., r\}$$

provided with the local coefficient system $W^{\lambda/\hbar}V^{-\lambda'/\hbar}$.

Due to Theorem 4.2 the above oscillating integral can be considered as the *mirror partner* of the concave toric bundle space E [19] in the sense of

GW-invariants in question. In fact such a relationship was the initial point of the whole project started by [13, 15] and completed in [12, 14].

Thus the two proofs of the same theorem appear to be variants of the same proof rather than two different ones, except that our reference to the general theory of equivariant quantum cohomology, developed in [12], Sections 1 – 6, for concave and convex vector bundles over convex manifolds, is replaced in [22] by a computation.

It is worth straightening some inaccuracy of [22] in quotation. As it is commonly known, "Givental's idea of studying equivariant Euler classes" (see p. 1 in [22]) is due to M. Kontsevich [19] who proposed a fixed point computation of such classes via summation over trees. The idea of the equivariant version of quantum cohomology listed on p. 6 of [22] among "a number of beautiful ideas introduced by Givental in [12, 13]" was actually suggested two years earlier in [16] by a different group of authors. The statement in the abstract that the paper [22] "is completing the program started by Candelas et al, Kontsevich, Manin and Givental, to compute rigorously the instanton prepotential function for the quintic in P^4" is also misleading: the paper is more likely to confirm that the program has been complete for two years.

[19]The series $I(q,\hbar)e^{(p \log q)/\hbar}$ is annihilated by any linear differential operator $\mathcal{D}(\hbar q \partial/\partial q, q, \hbar)$ which annihilates the integral with any cycle Γ, but usually not vice versa. In order to get a one-to-one correspondence here one should impose some constraint on the cycles. We do not know however an exact description of the corresponding homology group. Different choices of such a constraint should correspond to different toric bundles $E \to X$ whith the same matrices (m_{ij}) and (l_{ij}). Thus the integral formula itself (which depends only on these matrices) can represent mirror partners of several different spaces (depending on the level of the momentum map in the symplectic reduction procedure).

the generalized mirror conjecture suggested in [13]: the equivariant GW-potential $J^\beta(q,\hbar)e^{(p\log q/\hbar)}$ which plays the role of oscillating integrals in our symplectic topology — singularity theory dictionary coincides with the oscillating integral after the transformation to flat coordinates described in the theorem.

Regardless of the mirror theory one can use the integral representation in order to compute the genus 1 GW-potential of E via the Hessians Δ_α and the asymptotical coefficients R_α at the critical points of the phase function $\sum(W_j + \lambda_j \log W_j) + \sum(V_j - \lambda'_j \log V_j)$ under the constraints $\sum m_{ij} \log W_j - \sum l_{ij} \log V_j = \log q_i$, $i = 1, ..., r$. [20] We suggest the reader to recover the genus 1 potential $dG = dq/24q$ for $E = X = \mathbb{C}P^1$ by this method and observe that the asymptotical coefficients R_α coincide with $R^{(0)}_{\alpha\alpha} - c^\alpha_{-1}/12$.

Application. Consider a generic holomorphic sphere $\mathbb{C}P^1$ in a Calabi-Yau 3-fold. Such spheres occur in a discrete fashion with the normal bundle isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ which is concave. Multiple covers of this sphere contribute to genus 0 and 1 GW-potentials of the 3-fold, and the problem of computing these contributions reduces to studying GW-invariants of the non-compact total space E of the normal bundle.

According to Corollary 4.3 the hypergeometric series [21]

$$I = \sum_{d=0}^{\infty} q^d \frac{\Pi_{m=0}^{d-1}(p+m\hbar)^2}{\Pi_{m=1}^{d}(p-\lambda+m\hbar)(p+\lambda+m\hbar)}$$

coincides with the GW-potential J. The intersection index

$$\int_{[E]} \varphi(p)I = \frac{1}{2\pi i} \oint \varphi(p)I \frac{dp}{\lambda^2(p^2-\lambda^2)}$$

equals the sum of a $(d=0)$-term which has no limit at $\lambda = 0$ with a series which at $\lambda = 0$ turns into

$$\sum_{d=1}^{\infty} q^d \operatorname{Res}_{p=\infty} \frac{\varphi(p)dp}{p^2(p+d\hbar)^2}.$$

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[20]Notice that choosing $p_1, ..., p_r$ on the role of Lagrange multipliers we arrive at the equations of the critical points in the form $W_j = \sum_i p_i m_{ij} - \lambda_j$, $V_j = \lambda'_j - \sum_i p_i l_{ij}$.

[21]We reduce the group G here to the one-dimensional torus so that $H^*_G(\mathbb{C}P^1) = \mathbb{Q}[p,\lambda]/(p^2 - \lambda^2)$ where p is the equivariant Chern class of $\mathcal{O}(1)$.

This formula with $\varphi = \exp{(p \log q)}/\hbar$ determines the contribution of multiple covers of the sphere to the genus 0 GW-potentials $S_{\alpha\beta}$ of the Calabi-Yau 3-fold. The result coincides — and Corollary 4.3 explaines why — with the formula for such a contribution obtained in [15] by toric (and therefore heuristic) methods. It is shown in [15] how this result implies the famous formula [22] $D^3 Q^D/(1 - Q^D)$ for contributions of degree D spheres and their multiple covers to the Yukawa coupling $\langle P \circ P, P \rangle$ of a Calabi-Yau 3-fold.

Furthermore, according to Corollary 4.4 the class p satisfies the relation $p^2 = \lambda^2/(1 - q)$. in the equivariant quantum cohomology algebra of the concave space E. From this, we find the differentials of the canonical coordinates:

$$du_{\pm} = dt_0 + p_{\pm} d \log q = dt_0 \pm \lambda (1 - q)^{-1/2} d \log q$$

and thus $\partial/\partial u_+ - \partial/\partial u_- = (1 - q)^{1/2} \lambda^{-1} q \partial/\partial q$. From the intersection pairing $\langle 1, 1 \rangle = \langle p, p \rangle = 0$, $\langle 1, p \rangle = \lambda^{-2}$ in the equivariant cohomology of E we find the "Hessians": $1/\Delta_+ + 1/\Delta_- = 0$, $p_+/\Delta_+ + p_-/\Delta_- = \lambda^{-2}$ and thus

$$\Delta_{\pm} = \pm 2 \lambda^3 (1 - q)^{-1/2}.$$

In particular $d \log(\Delta_+ \Delta_-) = q(1 - q)^{-1} d \log q$. Using the expression of R_{ij} in terms of Δ_i we find, after some elementary computations,

$$dR_{++} = -dR_{--} = \frac{q^2 d \log q}{32 \lambda (1 - q)^{3/2}}$$

and therefore

$$R_{++} = -R_{--} = \frac{(1 - q)^{1/2}}{16 \lambda} + \frac{(1 - q)^{-1/2}}{16 \lambda} - \frac{1}{8 \lambda}.$$

Since the constants $-c_{-1}^{\pm}/12 = \pm 1/8\lambda$, we conclude that

$$(R_{++} - c_{-1}^+/12) du_+ + (R_{--} - c_{-1}^-/12) du_- = (\frac{1}{8} + \frac{1}{8(1 - q)}) d \log q.$$

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[22]The formula claimed by physicists [5] was first confirmed in [1] by toric methods and then rigorously justified by Yu. Manin, J. Bryan, R. Pandharipande by equivariant methods more elementary than the mirror theory. This example is also contained in [22] and [28].

Combining 1/2 of this with 1/48-th of $q(1-q)^{-1}d\log q$ we finally arrive at

$$dG = (\frac{1}{8} + \frac{q}{12(1-q)})d\log q.$$

In the application to counting multiple elliptic covers of $\mathbb{C}P^1$ the degree $d = 0$ term $1/8$ is to be ignored. The remaining part $d\log(1-q)^{-1/12}$ gives rise to the formula $\log(1-Q^D)^{-1/12}$ for the contribution of degree D rational curves to the genus 1 GW-potential of a Calabi-Yau 3-fold, claimed by physicists and recently confirmed by more elementary equivariant methods in [17].

5   Nonlinear Serre duality.

Let $Y \subset X$ be a submanifold given by a section of a vector bundle V. It is plausible that some GW-invariants of Y depend only on the bundle. In higher genus realization of this idea encounters some obstruction avoidable in the genus 0 case which we begin with.

Convex super-manifolds. The vector bundle $V : E \to X$ is called *convex* if it is spanned by global holomorphic sections. For a stable genus 0 map $f : \Sigma \to X$ we have $H^1(\Sigma, f^*V) = 0$ and thus the spaces $H^0(\Sigma, f^*V)$ form an orbi-bundle $V_{0,n,d} : E_{0,n,d} \to X_{0,n,d}$ over the moduli space. We introduce genus 0 GW-invariants of the *super-manifold* ΠE by defining the virtual fundamental class $[\Pi E_{0,n,d}]$ as the cap-product of the homology class $[X_{0,n,d}]$ with the cohomology class $Euler_G(V_{0,n,d})$. Here $Euler_G$ means the equivariant Euler class with respect to a (lifted to $V_{0,n,d}$) Hamiltonian action of G on $E \to X$ such that all fixed points are contained in the zero section. Therefore the GW-invariants $(A, B, ..., C)_n^d$ of the supermanifold ΠE take their values in $H^*(BG, \mathbb{Q})$.

The genus 0 equivariant GW-theory extends, as it is shown in [12], to super-manifolds without any serious changes and gives rise to a Frobenius structure over the ground ring $\mathbb{Q}(\lambda)[[\Lambda]]$ on the equivariant cohomology space $H = H_G^*(X, \mathbb{Q}[[\Lambda]])$. The Poincaré metric on H is induced by the pairing

$$\langle\phi, \psi\rangle := \int_{[X]} \phi \wedge \psi \wedge Euler_G(V).$$

The quasi-conformal structure is determined by the usual grading on $H_G^*(X)$ and by the 1-st Chern class of the super-manifold:

$$c_1(\mathcal{T}_{\Pi E}) := c_1(\mathcal{T}_X) - c_1(V).$$

The conformal dimension of the Frobenius structure equals $\dim_{\mathbb{C}} X - \dim_{\mathbb{C}}(V)$ which coincides with the super-dimension of ΠE.

The holomorphic section $s : X \to E$ restricted to a curve $f : \Sigma \to X$ induces an element in $H^0(\Sigma, f^*V)$ and thus — a section $s_{0,n,d} : X_{0,n,d} \to E_{0,n,d}$. The zero locus $s_{0,n,d}^{-1}(0)$ coincides with the moduli space $Y_{0,n,d}$ of stable maps to $Y = s^{-1}(0)$ of degree d *in the ambient space* X. The virtual fundamental class $[Y_{0,n,d}]$ in $X_{0,n,d}$ coincides with $[X_{0,n,d}] \cap Euler(V_{0,n,d})$. This (M. Kontsevich's) observation serves as a basis for applications of GW-theory of super-manifolds to complete intersections. It shows that in the non-equivariant limit $\lambda = 0$ the equivariant correlators $(A, B, ..., C)_n^d$ of ΠE turn into the corresponding correlators of Y *among classes induced from the ambient space* X.

Localization via materialization. Let us assume now that the group $G_{\mathbb{C}}$ is a torus acting on X with isolated zero- and one-dimensional orbits, that the bundle V is the sum of positive line bundles (so that it is convex and the dual bundle V^* is concave), and that the action is lifted to V and V^* in the dual fashion. We will show (following Section 12 in [12]) that the Frobenius structures of the convex super-manifold ΠE and of the concave bundle space E^* are closely related.

Let ϕ_α, as usually, be the basis of fixed points in $H_G^*(X)$. Denote e_α^0 and e_α' the Euler factors $Euler_G(T_\alpha X)$ and $Euler_G(V_\alpha) = (-1)^{\dim V} Euler_G(V^*)$ respectively. We put $s_\alpha^\beta(\hbar) := S_{\alpha\beta}(\hbar) e^{-u_\beta/\hbar} e_\beta^0 (e_\beta')^{-1}$ where u_β are materialized canonical coordinates of Theorem 3.2 applied to the case of the super-manifold ΠE.

Proposition 5.1. *The matrix* (s_α^β) *satisfies the recursion relation*

$$s_\alpha^\beta(\hbar) = \delta_{\alpha\beta} + \sum_{(\gamma,\beta)} \sum_{m=1}^{\infty} s_\alpha^\gamma(\chi_{\gamma\beta}/m) \, Coeff_\gamma^\beta(m) \frac{(q^{d_{\gamma\beta}} e^{(u_\gamma - u_\beta)/\chi_{\gamma\beta}})^m}{\hbar - \chi_{\gamma\beta}/m}.$$

The summation indices (γ, β) indicate one-dimensional orbits of $G_{\mathbb{C}}$ connecting the fixed points γ and β, and $d_{\gamma\beta}$ and $\chi_{\gamma\beta}$ have the same meaning as in Section 3.

Proof. The proposition is obtained by localization technique in the same way as the recursion relation in Step 1 in the proof of Theorem 4.2: we cut out the last edge (m-multiple cover of the orbit (γ, β)) in the chain connecting the vertices carrying ϕ_α and ϕ_β. The recursion coefficient takes in account the localization factor of the edge. However this time we use the localization factor of the last vertex (carrying ϕ_β) in the form described in Proposition 3.1: $v = (\hbar + \chi_{\beta\gamma}/m)^{-1} \exp(u_\beta/\hbar + mu_\beta/\chi_{\beta\gamma})$. $\square$

The linear recursion relation of Proposition 5.1 unambiguously determines the fundamental solution $(S_{\alpha\beta})$ as a function of canonical coordinates. The relation between canonical coordinates u_β and the flat coordinates t_β is non-linear and is obtained from the asymptotics $\sum_\alpha S_{\alpha\beta} = (1 + o(\hbar^{-1}))e^{t_\beta/\hbar}(e_\beta^0)^{-1}e_\beta'$. Expand $s_\alpha^\beta(\hbar)$ as $\delta_{\alpha\beta} + \dot{s}\,_\alpha^\beta \hbar^{-1} + o(\hbar^{-1})$. With this notation we arrive at

$$t_\beta = u_\beta + \sum_\alpha \dot{s}_\alpha^\beta.$$

Parallel results for concave bundles V^* look as follows. Denote here the fundamental solution matrix by $(S_{\alpha\beta}^*)$ and put $s^*\,_\alpha^\beta = (-1)^{\dim V} e_a' S_{\alpha\beta}^* e^{-u_\beta/\hbar} e_\beta^0$. Then $(s^*\,_\alpha^\beta(\hbar))$ satisfies the recursion relation of Proposition 5.1 with new recursion coefficients $Coeff^*\,_\gamma^\beta(m)$, where u_β are now the canonical coordinates of Theorem 3.1 applied to the concave bundle space E^*. Respectively, the flat coordinates t_β^* are found from $\sum_\alpha S_{\alpha\beta}^* = (1 + o(\hbar^{-1}))e^{t_\beta/\hbar}(e_\beta^0 e_\beta')^{-1}(-1)^{\dim V}$:

$$t_\beta^* = u_\beta + \sum_\alpha (e_\alpha')^{-1} \dot{s}^*\,_\alpha^\beta e_\beta'.$$

Now the Serre duality enters the game as the following identity:

$$Coeff^*\,_\gamma^\beta(m) = (-1)^{mc_{\gamma\beta}} Coeff_\gamma^\beta(m),$$

where $c_{\gamma\beta}$ is the value of the 1-st Chern class of the bundle V on the degree $d_{\gamma\beta}$ of the 1-dimensional orbit $(\gamma\beta)$.

Indeed, the coeefficients arise from fixed point localization formulas applied to the map $\varphi : \Sigma_0 \to \mathbb{C}P^1$, $z \mapsto w = z^m$ of $\Sigma_0 \simeq \mathbb{C}P^1$ onto the orbit. Both coefficients are ratios of two equivariant Euler classes which come from the virtual normal space to $X_{0,n,d}^G$ in $X_{0,n,d}$ (the denominators), and from the bundles $V_{0,n,d}$ and $V_{0,n,d}^{*'}$ respectively (the numerators). Due to

our choice of normalization for s and s^* the denominators are the same, and the numerators are the equivariant Euler classes respectively of the space (of holomorphic sections vanishing at $z = \infty$)

$$H^0(\Sigma_0, (\varphi^*V) \otimes \mathcal{O}_{\Sigma_0}(-[\infty]))$$

and of

$$H^1(\Sigma_0, (\varphi^*V^*) \otimes \mathcal{O}_{\Sigma_0}(-[0])).$$

By (elementary) Serre duality on Σ_0 the second space is canonically dual to

$$H^0(\Sigma_0, (\varphi^*V) \otimes \mathcal{K}_{\Sigma_0}([0])).$$

But the twisted canonical line bundle $\mathcal{K}_{\Sigma_0}([0] + [\infty])$ on $\Sigma_0 \simeq \mathbb{C}P^1$ is trivialized by the invariant section $d \log z$. Since $mc_{\gamma\beta}$ is the dimension of the dual cohomology spaces, we conclude that the numerators differ by the sign $(-1)^{mc_{\gamma\beta}}$.

Thus we arrive at the "nonlinear Serre duality" theorem [12].

Theorem 5.2. *The fundamental solution matrices $(S_{\alpha\beta})$ and $(S^*_{\alpha\beta})$ of the dual convex super-manifold ΠE and concave vector bundle space E^*, considered as functions of canonical coordinates, satisfy*

$$S_{\alpha\beta}(u, q, \hbar) = (-1)^{\dim V} e'_\alpha S^*_{\alpha\beta}(u, \pm q, \hbar) e'_\beta,$$

where $\pm q^d$ means $(-1)^{\langle c_1(V), d \rangle} q^d$.

In flat coordinates, the fundamental solution matrices are related therefore by an additional transformation of coordinates $t^* = t^*(u(t, q), \pm q)$. The transformation can be found directly from $(S^*_{\alpha\beta})$ in flat coordinates by comparing the asymptotics of row sums in Theorem 5.2 modulo $\hbar^{-2}$. Introduce $\dot{S}^*_{\alpha\beta}(t^*, q)$ by

$$S^*_{\alpha\beta} = [\delta_{\alpha\beta} + \dot{S}^*_{\alpha\beta} \hbar^{-1} + o(\hbar^{-1})] e^{t^*_\beta / \hbar} (e^0_\beta e'_\beta)^{-1} (-1)^{\dim V}.$$

After some elementary computation we get

$$t_\beta = t^*_\beta + \sum_\alpha e'_\alpha \dot{S}^*_{\alpha\beta}(t^*, \pm q)(e'_\beta)^{-1}.$$

Notice that modulo $\hbar^{-2}$ the GW-potential $S^*_{\alpha\beta}$ equals

$$\delta_{\alpha\beta}(e^0_\beta e'_\beta)^{-1}(-1)^{\dim V} + \hbar^{-1}\sum_{d,n} q^d(\phi_\alpha, t^*, ..., t^*, \phi_\beta)/n!.$$

We obtain from this that in more invariant terms the change of variables is describes by the GW-invariants

$$t_\beta = \sum_{n,d}(\pm q)^d(e', t^*, ..., t^*, \phi_\beta e^0_\beta)^*/n!$$

where $e' = \sum_\alpha e'_\alpha \phi_\alpha$ is the equivariant Euler class of V. In particular, the Jacobian of the change of variables is described by the operator $e'\circ$ of quantum multiplication in $H^*_G(E^*)$:

$$dt_\beta = \sum_\gamma(e\circ)^\beta_\gamma(t^*, \pm q)dt^*_\gamma.$$

We reiterate a question posed in [12]: how general is the nonlinear Serre duality relationship between GW-theory of dual supermanifolds and bundle spaces?

Toric supermanifolds. Let us assume now that V is a direct sum of *positive* line bundles over a toric symplectic manifold X with equivariant Chern classes

$$v_j = \sum_{i=1} p_i l_{ij} - \lambda'_j, \ \ j = 1, ..., l,$$

(p_i here are the same as in Theorem 4.2) and restrict ourselves to the study of the fundamental solution $(S_{\alpha\beta})$ for ΠE along $H^0 \oplus H^2$ (with coordinates t_0 and $t = \log q$ respectively). As in Section 4, we have

$$\sum_\alpha S_{\alpha\beta} = \langle J(q, \hbar)e^{(t_0 + p \log q)/\hbar}, \phi_\beta\rangle$$

where J is a formal vector q-series with coefficients which are rational functions of λ, λ' and $\hbar$. Combining the nonlinear Serre duality with the mirror theorem for V^* we conclude that the series J is obtained by a change of variables from its hypergeometric counterpart I. More precisely, introduce the hypergeometric vector-function

$$I = \sum_d q^d \Pi^l_{j=1}\frac{\Pi^{L_j(d)}_{k=-\infty}(v_j + k\hbar)}{\Pi^0_{k=-\infty}(v_j + k\hbar)}\Pi^m_{j=1}\frac{\Pi^0_{k=-\infty}(w_j + k\hbar)}{\Pi^{D_j(d)}_{k=-\infty}(w_j + k\hbar)}$$

whose terms differ from those in the hypergeometric series in Theorem 4.2 (let us denote here that series by I^*) by the factors $(-1)^l(v_1...v_l)^{-1}(v_1 + L_1(d)\hbar)...(v_l + L_l(d)\hbar)$ and by the signs $\pm q$. Taking into account Corollaries 4.3 and 4.4 we arrive at the following mirror theorem for toric supermanifolds [14]. [23]

Corollary 5.3. *Suppose that the toric supermanifold* ΠE *has non-negative 1-st Chern class and that* $\dim V > 1$*. Then the GW-potential* $J(q, \hbar)e^{(t_0+p\log q)/\hbar}$ *is obtained from the hypergeometric vector series* $I(q, \hbar)e^{(t_0+p\log q)/\hbar}$ *by the division* $I \mapsto I/\varphi(q)$ *and by the change of variables*

$$t_0 \mapsto t_0 + \lambda g(q) + \lambda' g'(q) + f_0, \ \log q_i \mapsto \log q_i + f_i(q)$$

unambiguously determined by the asymptotics

$$I = \varphi[1 + \hbar^{-1}(\lambda g + \lambda' g' + f_0 + f_1 p_1 + ... + f_r p_r) + o(\hbar^{-1})].$$

Proof. The series $v_1...v_l I e^{(p\log q)/\hbar}$ is actually obtained by differentiating $I^* e^{(p\log q)/\hbar}$, as in Corollary 4.4, in the directions corresponding to the classes [24] $-v_1, ..., -v_l$ and changing q to $\pm q$. According to Corollary 4.4 the product $e' = v_1 \circ ... \circ v_l$ in the quantum cohomology algebra of E^* equals $v_1...v_l \varphi(\pm q)$. This identifies components of the ratio $\varphi^{-1} I e^{(t_0+p\log q)/\hbar}$ with the GW-potentials

$$\sum(\pm q)^d (e', \frac{e^{(t_0+p\log q)/\hbar}\phi_\beta e_\beta^0}{\hbar - c})^*$$

whose asymptotical terms of order $\hbar^{-1}$ determine the change of variables prescribed by Theorem 5.2.

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[23]Our hypotheses here are somewhat more restrictive than in [14], first — because we assume that V is *strictly* positive, and second — because of the condition $\dim V > 1$. In applications to hypersurfaces, say, in $\mathbb{C}P^n$ the last condition is not constraining because one can describe the same hypersurface as a codimension 2 complete intersection in $\mathbb{C}P^{n+1}$. (We suggest the reader to consider the example of Calabi-Yau 3-folds given by two equations of degrees 5 and 1 in $\mathbb{C}P^5$ in order to observe how the mirror transformation of [5] emerges from our formulas in the non-equivariant limit.) We believe that the same trick can be applied to hypersurfaces in general toric varieties by extending GW-theory to Kähler orbifolds following M. Kontsevich's proposal.

[24]Notice that v_j here correspond to $-v_j$ in Section 4.

Elliptic GW-invariants. Trying to extend GW-theory of convex syper-manifolds to higher genera in a way consistent with GW-theory for corresponding complete intersections we encounter the following difficulty. Let $f : (\Sigma, \varepsilon) \to X$ be a stable map. Even if the bundle $V : E \to X$ is convex, the space $H^1(\Sigma, f^*V)$ can be nontrivial depending on the map. As a result, the spaces $H^0(\Sigma, f^*V)$ do not form a vector bundle over $X_{g,n,d}$. In any way, for a complete intersection Y given by a section of V the virtual fundamental class $[Y_{g,n,d}]$ in $X_{g,n,d}$ does not have to be the cap-product of a cohomology class with the virtual fundamental class $[X_{g,n,d}]$.

Counter-example. Consider the moduli space $X_{1,1,0} = X \times \overline{\mathcal{M}}_{1,1}$ of degree 0 elliptic maps to an m-dimensional X. As we know, the virtual fundamental class is Poincaré-dual to $c_m(X) - \omega \, c_{m-1}(X)$. On the other hand the virtual fundamental class $[Y_{1,1,0}]$ for an (m-l)-dimensional submanifold $i : Y \subset X$ is determined by the push-forwards $i_*(c_{m-l-1}(Y))$ and $i_*(c_{m-l}(Y))$. When Y is given by a section of V, the push-forwards can be computed in terms of V. Namely, the section identifies the normal bundle to Y in X with i^*V. Thus the (unstable) total Chern class of Y equals

$$\xi^{m-l} + c_1(Y)\xi^{m-l-1} + ... + c_{m-l}(Y) = i^* \frac{\xi^m + c_1(X)\xi^{m-1} + ... + c_m(X)}{\xi^l + c_1(V)\xi^{l-1} + ... + c_l(Y)}.$$

The Chern classes of Y we need are extracted from this ratio as

$$c_{m-l}(Y) = i^* \operatorname{Res}_{\xi=\infty} \frac{Chern(X)}{Chern(V)} \frac{d\xi}{\xi} \text{ and } c_{m-l-1}(Y) = \operatorname{Res}_{\xi=\infty} \frac{Chern(X)}{Chern(V)} \frac{d\xi}{\xi^2}.$$

Since $\omega^2 = 0$, the virtual fundamental class $i_*(c_{m-l}(Y) - \omega \, c_{m-l-1}(Y)$ of $[Y_{1,1,0}]$ is therefore computed as the Poincaré-dual to

$$Euler(V) \operatorname{Res}_{\xi=\infty} \frac{Chern(X)}{Chern(V)} \frac{d\xi}{\xi + \omega}.$$

and has no reason to be a multiple of $c_m(X) - \omega \, c_{m-1}(X)$.

We believe that the virtual fundamental class $[Y_{g,n,d}]$ in $X_{g,n,d}$ is the non-equivariant limit of a suitable equivariant homology class in $X_{g,n,d}$ and thus can be expressed via localization formulas in terms of the fundamental class $[X_{g,n,d}^G]$ of the fixed point orbifold. This would allow one to include complete intersections in the domain of applications of the results and methods of this paper. We are not ready however to report upon any progress in this direction and hope to return to this problem elsewhere.

References

[1] P. Aspinwall, D. Morrison. *Topological field theory and rational curves.* Commun. Math. Phys. **151** (1993), 245 – 262.

[2] K. Behrend. *Gromov – Witten invariants in algebraic geometry.* Invent. Math. **127** (1997), 601 – 617.

[3] K. Behrend, B. Fantechi. *The intrinsic normal cone.* Invent. Math. **128** (1997), 45 – 88.

[4] K. Behrend, Yu. Manin. *Stacks of stable maps and Gromov - Witten invariants.* Duke Math. J. **85** (1996), 1 – 60.

[5] P. Candelas, X. C. de la Ossa, P. S. Green, L. Parkes. *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory.* Nucl. Phys. B **359** (1991), 21 – 74.

[6] R. Dijkgraaf. *Mirror symmetry and elliptic curves.* Progr. in Math. **129**, Birkhäuser, 1995.

[7] B. Dubrovin. *The geometry of $2D$ topological field theories.* In: *Integrable systems and quantum groups.* Lecture Notes in Math. **1620**, Springer-Verlag, Berlin, 1996, 120 – 348.

[8] B. Dubrovin, Y. Zhang. *Bihamiltonian hierarchies in $2D$ topological field theory at one-loop approximation.* Preprint, hep-th/9712232.

[9] K. Fukaya, K. Ono. *Arnold conjecture and Gromov - Witten invariant.* Preprint, 1996.

[10] E. Getzler. *Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants.* Preprint, alg-geom/9612004.

[11] E. Getzler. *Topological recursion relation in genus* 2. Preprint, alg-geom/9801003.

[12] A. Givental. *Equivariant Gromov - Witten invariants.* Internat. Math. Res. Notices **1996**, No. 13, 613 – 663.

[13] A. Givental. *Homological geometry and mirror symmetry.* In: *Proceedings of the International Congress of Mathematicians, 1994, Zürich,* Birkhäuser, Basel, 1995, 472 – 480.

[14] A. Givental. *A mirror theorem for toric complete intersections.* Preprint, alg-geom/9701016.

[15] A. Givental. *Homological geometry I: Projective hypersurfaces.* Selecta Math. (N. S.) **1** (1995), 325 – 345.

[16] A. Givental, B. Kim. *Quantum cohomology of flag manifolds and Toda lattices.* Commun. Math. Phys. **168** (1995), 609 – 641.

[17] T. Graber, R. Pandharipande. *Localization of virtual classes.* Preprint, alg-geom/9708001.

[18] B. Kim. *On equivariant quantum cohomology.* Internat. Math. Res. Notices **1996**, No. 17.

[19] M. Kontsevich. *Enumeration of rational curves via toric actions.* In: *The moduli space of curves,* ed. by R. Dijkgraaf, C. Faber, G. van der Geer. Progr. in Math. **129**, Birkhäuser, Boston, 1995, 335 – 368.

[20] M. Kontsevich, Yu. Manin. *Gromov-Witten classes, quantum cohomology and enumerative geometry.* Commun. Math. Phys. **164** (1994), 525 – 562.

[21] J. Li, G. Tian. *Virtual moduli cycles and Gromov - Witten invariants in general symplectic manifolds.* Preprint, alg-geom/9608032.

[22] B. Lian, K. Liu, S.-T. Yau. *Mirror principle I.* Preprint, alg-geom/9712011.

[23] P. Lu. *A rigorous definition of fiberwise quantum cohomology and equivariant quantum cohomology.* Preprint.

[24] Y. Ruan. *Virtual neighborhoods and pseudo-holomorphic curves.* Preprint, alg-geom/9611021.

[25] K. Saito. *On periods of primitive integrals, I.* Preprint 412, Res. Inst. Math. Sci., 1982.

[26] B. Siebert. *Gromov – Witten invariants for general symplectic manifolds.* Preprint, dg-ga/9608005.

[27] B. Siebert. *Symplectic Gromov – Witten invariants.* To appear in: Proceedings of the AGE conference, Warwick 1996, K. Hulek, M. Reid (eds.), Cambridge Univ. Press.

[28] *Topics in Enumerative Algebraic Geometry.* Lecture notes based on the course taught by the author at UC Berkeley in Fall 97, in preparation.

[29] E. Witten. *Two-dimensional gravity and intersection theory on the moduli space.* Surveys in Diff Geom. **1** (1991), 243 – 310.