SOLITON EQUATIONS, VERTEX OPERATORS, AND SIMPLE SINGULARITIES

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Abstract. We prove the equivalence of two hierarchies of soliton equations associated to a simply-laced finite Dynkin diagram. The first was defined by Kac and Wakimoto [12] using the principal realization of the basic representations of the corresponding affine Kac–Moody algebra. The second was defined in [7] using the Frobenius structure on the local ring of the corresponding simple singularity. We also obtain a deformation of the principal realization of the basic representation over the space of miniversal deformations of the corresponding singularity. As a by-product, we compute the operator product expansions of pairs of vertex operators defined in terms of Picard–Lefschetz periods for more general singularities. Thus, we establish a surprising link between twisted vertex operators and deformation theory of singularities.

1. Introduction

The principal hierarchy of soliton equations associated to an affine Kac–Moody algebra of type $X_N^{(1)}$, where $X = ADE$, has been defined by V. Kac and M. Wakimoto [12] as the following systems of Hirota bilinear equations:

$$\operatorname{Res}\frac{d\zeta}{\zeta}\Big(\sum_{i=1}^{N} a_i\Gamma^{\alpha_i}\otimes\Gamma^{-\alpha_i}\Big)(\tau\otimes\tau) = h^{-2}(\rho|\rho)(\tau\otimes\tau) + h^{-1}\Big(\sum_{m\in E_+} m(y_m\otimes 1 - 1\otimes y_m)(\partial_{y_m}\otimes 1 - 1\otimes\partial_{y_m})\Big)(\tau\otimes\tau). \tag{1}$$

Here $\Gamma^{\pm\alpha_i}$ are vertex operators

$$\Gamma^{\pm\alpha_i} = \exp\Big(\pm\sum_{m\in E_+}\beta_{i,m}\, y_m\,\zeta^m\Big)\exp\Big(\mp\sum_{m\in E_+}\beta_{i,-m}\,\partial_{y_m}\,\zeta^{-m}/m\Big), \tag{2}$$

acting on a certain Fock space $\mathbb{C}[y_m, m\in E_+]$. The Fock space and the operators are constructed from the following data associated to a root system X_N of ADE type:

- M is a *Coxeter transformation* of the root system.
- α_i, $1\le i\le N$, are roots chosen one from each orbit of M on the set of roots.
- h is the Coxeter number (i.e., the order of M).

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- m_a are the Coxeter exponents (in particular, this means that $e^{2\pi i m_a/h}$, $1 \le a \le N$, are the eigenvalues of M), ordered, so that $m_a \le m_b$ when $a < b$.
- $E_+ = \{(a,n) \mid 1 \le a \le N,\ n \in \mathbb{Z}_{\ge 0}\}$. Abusing notation, we will write $m \in E_+$ for $m = m_a + nh$. (In all cases except $X_N = D_N$ with N even this notation is unambiguous since it embeds E_+ as a subset into $\mathbb{Z}_{>0}$.)
- $\beta_{i,m} = \alpha_i(H_{a(m)})$, where the subscript $a(m)$ is defined by $m = m_{a(m)} + nh$, and $\{H_a \mid a = 1,\dots,N\}$ is an eigenbasis of M satisfying the normalization condition $(H_a, H_b) = h\delta_{a+b,N+1}$. Here $(\cdot \mid \cdot)$ is the invariant inner product normalized by the condition $(\alpha \mid \alpha) = 2$ for all roots α.
- $(\rho \mid \rho) = Nh(h+1)/12$ is the value of the inner square of the sum ρ of fundamental weights (see, e.g., [10]).

The coefficients a_i are defined in terms of the principal realization of the basic representation (see [12] and Section 2 below).

On the other hand, an *a priori* different hierarchy was constructed in [7] for each simple singularity of ADE type using the Picard–Lefschetz periods of the singularity. Moreover, it was shown in [7] that the equations of this hierarchy have the same form as (1) (for the corresponding root system of ADE type) except that the coefficients, which we will now denote $\widetilde{a}_i$ (instead of a_i) are *a priori* different. These coefficients, which are actually defined in terms of certain limits associated with the singularity, were characterized in [7] by the following conditions:

$$\widetilde{a}_i/\widetilde{a}_j = \prod_\alpha (H_1 \mid \alpha)^{-(\alpha_i \mid \alpha)^2/2} \Big/ \prod_\alpha (H_1 \mid \alpha)^{-(\alpha_j \mid \alpha)^2/2}, \quad \text{and} \quad \sum_{i=1}^N \widetilde{a}_i = h^{-2}(\rho \mid \rho). \tag{3}$$

It was conjectured in [7] that the two hierarchies coincide (in other words, $\widetilde{a}_i = a_i$ for all i), and the conjecture was partially verified in [7] and [15]. However, it was left open in general; one of the problems was that the Kac–Wakimoto coefficients a_i were not given in all cases.

In this paper we prove the following result for all ADE types:

Theorem. *The two hierarchies coincide; namely,*

$$a_i = \widetilde{a}_i \quad \text{for all } i = 1,\dots,N.$$

Our proof is based on computing explicitly the operator product expansions (OPEs) for the vertex operators defined in [7] in the context of singularity theory and matching them with the OPEs of the vertex operators of representation theory. We then use these OPEs to prove that $a_i = \widetilde{a}_i$. Actually, the vertex operators defined from singularity theory and their OPEs make sense for singularities much more general than the ADE singularities, and in a sense the main goal of this paper is to expose their place in representation theory.

In fact, it turns out that a similar result holds in the case of more general singularities. In this case the role of an affine Kac–Moody algebra is played by the lattice vertex algebra associated to the Milnor lattice of the singularity (the middle cohomology of the generic fiber of the singularity equipped with the intersection pairing). Using the Picard–Lefschetz periods associated to the singularity, we define twisted vertex operators which give rise to twisted modules over this lattice vertex algebra (with respect to the automorphism of the lattice given by the monodromy) in a similar fashion to what is described in this paper in the case of simple singularities. This will be discussed in more detail in a subsequent paper.

Thus, we prove that the Kac–Wakimoto hierarchy, which is constructed using the basic representation of the affine Kac–Moody algebra $X_N^{(1)}$, where $X = ADE$, and the hierarchy of [7], which is attached to the singularity of type X_N, coincide. However, the latter is naturally included in a family of hierarchies parametrized by the space of miniversal deformations of the singularity, using the Frobenius structure of the singularity. In Section 4 we will use this structure to produce a *family* of representations of $X_N^{(1)}$ defined by certain deformations $\Gamma^\alpha_\tau(\zeta)$ of the vertex operators $\Gamma^\alpha(\zeta)$ introduced above, where τ lies in the space of miniversal deformations of the singularity of type X_N. These deformations are given in terms of the Picard–Lefschetz periods associated to simple singularities. Thus, we obtain a surprising link between representations of affine Kac–Moody algebras of ADE type and simple singularities.

Although this is not used in our proof of the above theorem, we also obtain an explicit formula for the coefficients a_i for all simply-laced affine Kac–Moody algebras:

$$a_i = h^{-1} \prod_{k=1}^{h-1} \Big(1-\eta^k\Big)^{(\alpha_i\,|\,M^k\alpha_i)}, \qquad \eta = e^{2\pi\sqrt{-1}/h}. \tag{4}$$

Formulas (3), (4) and the theorem then imply that the following ratios are the same for all roots α_i:

$$\prod_{k=1}^{h-1} \Big(1-\eta^k\Big)^{(\alpha_i\,|\,M^k\alpha_i)} \Big/ \prod_{\alpha} (H_1 \;|\,\alpha)^{-(\alpha_i\,|\,\alpha)^2/2}.$$

While it should not be hard to verify this on the case-by-case basis, it is an interesting question whether one can find a general direct proof of this fact, which expresses the proportionality between $\{a_i\}$ and $\{\widetilde{a}_i\}$. Since $\sum a_i$ (which can be easily found from the consistency condition on the Kac–Wakimoto hierarchy) coincides with $\sum \widetilde{a}_i$, this would provide another, more elementary, proof of our theorem.

The paper is organized as follows. In Section 2, we recall the construction of hierarchies of Kac–Wakimoto and derive formula (4).

In Section 3, we recall from [7] the construction of vertex operators based on Frobenius structures of singularity theory, and in the case of simple singularities, provide a uniform identification of the coefficients $\widetilde{a}_i$ with their counterparts in the Kac–Wakimoto theory for all ADE types.

In Section 3, we already use *families* of vertex operators parametrized by the miniversal deformation of a (simple) singularity. In Section 4, we show that they define a family of realizations of the basic representation of level 1 of the corresponding affine Kac–Moody algebra. In fact, the intertwiners between the realizations of this family have already been constructed in [7].

2. The Kac–Wakimoto hierarchy

In abstract terms, the Kac–Wakimoto hierarchy describes the points of the Grassmannian which is the orbit of the highest weight vector in the level 1 basic representation (more precisely, in its principal realization) of the affine Kac–Moody algebra under the action of the corresponding group.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of the type $X_N = A_N, D_N$, or E_N. By definition, the affine Kac–Moody algebra corresponding to $\mathfrak{g}$ is the vector space

$$\widehat{\mathfrak{g}} := \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}\,K \oplus \mathbb{C}\,d$$

equipped with the Lie bracket defined by the following relations:

$$[X\ t^n, Y\ t^m] := [X,Y]\ t^{n+m} + n\delta_{n,-m}(X\,|\,Y)K,$$
$$[d, X\ t^n] := n(X\ t^n), \quad [K,\widehat{\mathfrak{g}}] := 0.$$

Here $X, Y \in \mathfrak{g}$, and $(\cdot \mid \cdot)$ denotes the adjoint-invariant bilinear form on $\mathfrak{g}$ normalized so that $(\alpha\,|\,\alpha) = 2$ for all roots α.

The principal realization of the level 1 basic representation depends on the choice of a Coxeter element M. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$, and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

its root decomposition. Extend the action of M on $\mathfrak{h}$ to a finite order inner automorphism of $\mathfrak{g}$ in the standard way. Under the action of M on $\mathfrak{h}$ all roots form N orbits of cardinality h, and the corresponding root spaces are likewise permuted by M: $M \cdot \mathfrak{g}_\alpha = \mathfrak{g}_{M\alpha}$. We pick generators $A_\alpha \in \mathfrak{g}_\alpha$ in such a way that they form an M-invariant set of vectors. We fix representatives $\alpha_1, \dots, \alpha_N$ in the M-orbits on the root system. Put $\eta = e^{2\pi i/h}$. Using this notation, we lift the action of M to $\widehat{\mathfrak{g}}$:

$$M(X\ t^m) = MX\ (\eta^{-1}t)^m, \quad MK = K, \quad Md = d.$$

We will frequently use the projector $x \mapsto h^{-1}\sum_{k=1}^h M^k x$ to the Lie subalgebra $\widehat{\mathfrak{g}}^M$ of M-invariant vectors. It is an important, and non-trivial, fact (see [10] for a proof) that *the subalgebra $\widehat{\mathfrak{g}}^M$ is isomorphic to the Lie algebra $\widehat{\mathfrak{g}}$ itself.* The principal construction of the basic representation is based on the property of M-invariant space

$$\mathfrak{h}[t,t^{-1}]^M \oplus \mathbb{C}K$$

to be a Lie subalgebra isomorphic to the Heisenberg Lie algebra. An important result of [9] (see also [10]) is that *the standard level 1 Fock representation of this Heisenberg algebra extends uniquely to a representation of $\widehat{\mathfrak{g}}^M$.*

A vector v in the orbit of the highest weight vector in this representation under the action of the corresponding group may be characterized by the property that $v \otimes v$ is an eigenvector, with a certain specific eigenvalue, of the bilinear Casimir operator. Introduce a basis $H_1, \dots, H_N$ of $\mathfrak{h}$ formed by eigenvectors of M with the eigenvalues η^{m_i}, ordered so that $m_i \le m_j$ when $i < j$, and normalized by the condition

$$(H_i|H_j) = h\delta_{i+j,N+1}.$$

Introduce the notation

$$H_{i,m} = h^{-1} \sum_{k=1}^{h} M^k(H_i\ t^m), \quad \text{and} \quad A_{\pm\alpha_i,m} = h^{-1} \sum_{k=1}^{h} M^k(A_{\pm\alpha_i}\ t^m), \quad i = 1, \dots, N.$$

The elements $K, d, A_{\alpha_i,m}$ with $m \in \mathbb{Z}$, and $H_{i,m}$ with $m \equiv m_i \mod h$ (note that all other $H_{i,m} = 0$) form a basis in the Lie algebra $\widehat{\mathfrak{g}}^M$. Restricting an invariant inner product from $\widehat{\mathfrak{g}}$ to $\widehat{\mathfrak{g}}^M$ and computing it in this basis, it is not hard to see that the following element of $\widehat{\mathfrak{g}}^M \otimes \widehat{\mathfrak{g}}^M$ commutes with the diagonal action of $\widehat{\mathfrak{g}}^M$:

$$\sum_{i,m} \frac{h}{(A_{\alpha_i} \,|\, A_{-\alpha_i})} A_{\alpha_i,m} \otimes A_{-\alpha_i,-m} + \frac{1}{h} \sum_{\substack{i+j=N+1 \\ m \equiv m_i \mod h}} H_{i,m} \otimes H_{j,-m} \ + \ K \otimes d + d \otimes K.$$

This is the bilinear Casimir element.

Consider the representation of the Heisenberg Lie subalgebra $\mathfrak{h}[t, t^{-1}]^M \oplus \mathbb{C}K$ on the Fock space $\mathbb{C}[y_m \,|\, m \in E_+]$ given by the formulas

$$K \mapsto 1/h, \quad \text{and} \quad H_{i,\pm m} \mapsto \begin{cases} \partial/\partial y_m, \\ m y_m \end{cases} \text{for } m \in E_+.$$

According to [9], this representation of the Heisenberg Lie subalgebra on the Fock representation extends to the level 1 basic representation of $\widehat{\mathfrak{g}}^M$. From the commutation relations of d with $H_{i,m}$, it follows immediately that under this action[1]

$$d \mapsto -\sum m y_m \partial/\partial y_m,$$

the Euler vector field for the grading $\deg y_m = -m$ on the Fock space.

Following [12], we use generating functions

$$x_{\pm\alpha_i}(\zeta) := \sum_m A_{\pm\alpha_i,m} \zeta^{-m}.$$

It is easy to check that when $m \equiv m_i \mod h$,

$$[H_{i,m}, x_{\pm\alpha_j}(\zeta)] = \pm\alpha_j(H_i)\zeta^{-m} x_{\pm\alpha_j}(\zeta).$$

This commutation relation coincides with the one between the operators representing $H_{i,m}$ in the Fock space and the vertex operators

$$\Gamma^{\pm\alpha_i} = \exp\Big(\pm \sum_{m \in E_+} \alpha_i(H_{a(m)})\, y_m\, \zeta^m\Big) \exp\Big(\mp \sum_{m \in E_+} \alpha_i(H_{a(-m)})\, \partial_{y_m}\, \zeta^{-m}/m\Big). \tag{5}$$

Moreover, according to a general lemma from [10], this implies that $x_{\pm\alpha_i}$ is proportional to $\Gamma^{\pm\alpha_i}$. Putting $\deg \zeta = -1$, we make both the vertex operators and the generating functions homogeneous of degree 0 with respect to the grading in the Fock space defined by d. This implies that the coefficient of proportionality does not depend on ζ.

Define a_i by the formula

$$\frac{h}{(A_{\alpha_i} \,|\, A_{-\alpha_i})} x_{\alpha_i}(\zeta) \otimes x_{-\alpha_i}(\zeta) = a_i \Gamma^{\alpha_i}(\zeta) \otimes \Gamma^{-\alpha_i}(\zeta). \tag{6}$$

[1] Up to an additive constant, which can be made 0 by redefining d as $d - \mathit{const} \cdot K$.

Noting that

$$\mathrm{Res}\ \frac{d\zeta}{\zeta} x_{\alpha_i}(\zeta) \otimes x_{-\alpha_i}(\zeta) = \sum_m A_{\alpha_i,m} \otimes A_{-\alpha_i,-m},$$

we express the bilinear Casimir operator in our representation as follows:

$$\mathrm{Res}\,\frac{d\zeta}{\zeta} \sum_i a_i \Gamma^{\alpha_i}(\zeta) \otimes \Gamma^{-\alpha_i}(\zeta) \ - \sum_{m\in E_+} \frac{m}{h}(y_m \otimes 1 - 1 \otimes y_m)(\partial_{y_m} \otimes 1 - 1 \otimes \partial_{y_m}).$$

The eigenvalue of this operator on tensor squares $v \otimes v$, where v is the orbit of the highest weight vector under the action of the corresponding group, may be computed using an explicit (and non-trivial) identification $\widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}^M$. We are not going to reproduce this computation here, but merely quote the answer (see [10]): $(\rho\,|\,\rho)/h^2$. Denoting by τ the element of the Fock representation corresponding to v, we arrive at the Hirota bilinear equation (1) from the Introduction.

We now wish to compute the coefficients a_i appearing in formula (6).

Lemma 1. *The expression*

$$(\zeta^h - w^h)\left(\frac{x_{\alpha_i}(\zeta)}{\zeta}\frac{x_{-\alpha_i}(w)}{w} - \frac{(A_{\alpha_i}\,|\,A_{-\alpha_i})}{h}\frac{K}{(\zeta-w)^2}\right),$$

expanded in the region $|w| < |\zeta|$ into a formal power series in $\zeta^{\pm 1}$ and $w^{\pm 1}$ with coefficients which are operators in our Fock space, has a well-defined limit as $\zeta \to w$.

Proof. Introduce the *normal ordering* by the formula

$$:A_{\alpha_i,n}A_{-\alpha_i,l}:= \begin{cases} A_{\alpha_i,n}A_{-\alpha_i,l} & \text{if } n<0 \\ A_{-\alpha_i,l}A_{\alpha_i,n} & \text{if } n \geq 0. \end{cases} \tag{7}$$

Then the formal power series $:x_{\alpha_i}(\zeta)x_{-\alpha_i}(w):$ has a well-defined limit as $\zeta \to w$ as an operator acting on individual elements of the Fock space. We compute the difference

$$x_{\alpha_i}(\zeta)x_{-\alpha_i}(w) - :x_{\alpha_i}(\zeta)x_{-\alpha_i}(w): = \sum_{m\in\mathbb{Z}}\sum_{n\geq 0}[A_{\alpha_i,n}, A_{-\alpha_i,m}]\zeta^{-n}w^{-m}.$$

We have

$$[A_{\alpha_i,n}, A_{-\alpha_i,m}] = \frac{(A_{\alpha_i}\,|\,A_{-\alpha_i})}{h} n\delta_{m+n,0}K \ + \ h^{-2}\sum_{k,l=1}^{h}[M^l A\alpha_i, M^k A_{-\alpha_i}]\eta^{-nl-mk}t^{n+m}.$$

Denoting $[M^l A_{\alpha_i}, A_{-\alpha_i}]$ by B_i^l, we can rearrange contributions of second summands as

$$h^{-2}\sum_{m\in\mathbb{Z}}\sum_{n\geq 0}\zeta^{-n}w^{-m}\sum_{l=1}^{h}\eta^{-ln}\sum_{k=1}^{h}M^k B_i^l \eta^{-k(n+m)}t^{n+m} = \frac{1}{h}\sum_{l=1}^{h}\frac{\eta^l\zeta}{\eta^l\zeta - w}\sum_{m\in\mathbb{Z}}B_{i,m}^l w^{-m}.$$

The summands have pole at $w = \eta^l\zeta$ of order at most 1, and hence have a well-defined limit after multiplication by $\zeta^h - w^h$. The first summands of the commutators add up to

$$\frac{(A_{\alpha_i}\,|\,A_{-\alpha_i})}{h}\sum_{n\geq 0} n\left(\frac{w}{\zeta}\right)^n K = \frac{\zeta w}{(\zeta-w)^2}\frac{(A_{\alpha_i}\,|\,A_{-\alpha_i})}{h}K.$$

$\square$

The multiplication and differentiation parts of the vertex operators (5) are elements of a Heisenberg group. Such operators commute up to a scalar factor. Define $B_i(\zeta, w)$ by the formula (OPE)

$$\Gamma^{\alpha_i}(\zeta)\Gamma^{-\alpha_i}(w) = B_i(\zeta, w) :\Gamma^{\alpha_i}(\zeta)\Gamma^{-\alpha_i}(w): , \tag{8}$$

where the normal ordering on the RHS is defined by moving all differentiation operators $\partial/\partial y_m$ to the right (i.e., by taking the commutator of the differentiation part of the vertex operator on the left with the multiplication part of the vertex operator on the right).[2] The normally ordered product on the RHS has a well-defined limit as $\zeta \to w$, and moreover, this limit is obviously equal to 1.

Corollary 2.

$$a_i^{-1} = \lim_{\zeta \to w} (1 - w/\zeta)(1 - w^h/\zeta^h) B_i(\zeta, w),$$

Proof. We have

$$(\zeta - w)(\zeta^h - w^h)\Gamma^{\alpha_i}(\zeta)\Gamma^{-\alpha_i}(w) = (\zeta - w)(\zeta^h - w^h) B_i(\zeta, w) : \Gamma^{\alpha_i}(\zeta)\Gamma^{-\alpha_i}(w) :$$

$$= \frac{h a_i^{-1}(\zeta - w)(\zeta^h - w^h)}{(A_{\alpha_i} \,|\, A_{-\alpha_i})} x_{\alpha_i}(\zeta) x_{-\alpha_i}(w) = a_i^{-1} \zeta w \frac{\zeta^h - w^h}{\zeta - w} K + (\zeta - w)(\text{regular terms}).$$

Passing to the limit $\zeta \to w$ and using Lemma 1 and the fact that $K \mapsto 1/h$ in our representation, we obtain the desired result. $\square$

Lemma 3.

$$B_i(\zeta, w) = \prod_{k=1}^{h} \Big(1 - \eta^k \, \frac{w}{\zeta}\Big)^{-(\alpha_i \,|\, M^k \alpha_i)}, \tag{9}$$

Proof. The factor $B_i(\zeta, w)$ in (8) is obtained by commuting the second and the first exponential factors, respectively, of Γ^{α_i} and $\Gamma^{-\alpha_i}$, i.e.,

$$B_i(\zeta, w) = \exp\Big(\sum_{m \in E_+} \alpha_i(H_{a(-m)}) \alpha_i(H_{a(m)}) \frac{(w/\zeta)^m}{m} \Big). \tag{10}$$

Consider the projection $h^{-1} \sum_{k=1}^{h} \eta^{mk} M^k \alpha_i$ of α_i to the eigenspace of M in $\mathfrak{h}^*$ with the eigenvalue η^{-m}. Since the eigenspaces are pairwise orthogonal and the bases $\{H_a\}$ and $\{h^{-1} H_{N+1-a}\}$ formed by the eigenvectors are dual, the projection of α_i can be written as $h^{-1}(\alpha_i \,|\, H_{a(m)}) H_{a(-m)}$. Pairing this with α_i, we find that

$$\alpha_i(H_{a(-m)}) \alpha_i(H_{a(m)}) = \sum_{k=1}^{h} \eta^{mk} (\alpha_i \,|\, M^k \alpha_i),$$

[2] Note that this normal ordering is different from the one defined by formula (7).

Note that this identity extends to $m \notin E_+$, since the RHS is equal to zero in this case. Therefore

$$\ln B_i = \sum_{k=1}^{h} (\alpha_i \,|\, M^k \alpha_i) \sum_{m=1}^{\infty} \frac{(\eta^k w/\zeta)^m}{m} = -\sum_{k=1}^{h} (\alpha_i \,|\, M^k \alpha_i) \ln(1 - \eta^k \frac{w}{\zeta}).$$

□

Combining Lemma 3 and Corollary 2, we obtain

Corollary 4.

$$a_i = h^{-1} \prod_{k=1}^{h-1} \left(1 - \eta^k\right)^{(\alpha_i \,|\, M^k \alpha_i)}. \tag{11}$$

The first equation of the Kac–Wakimoto hierarchy reads

$$\sum_{i=1}^{N} a_i = h^{-2} (\rho|\rho),$$

which coincides with the value of $\sum_{i=1}^{N} \widetilde{a}_i$ (see formula (3)). Hence in order to prove that $a_i = \widetilde{a}_i$ it suffices to check that

$$\frac{a_i}{a_j} = \frac{\widetilde{a}_i}{\widetilde{a}_j},$$

where the right hand side is given by formula (3). This has been done for A_N, D_4 and E_6 in [7] and for D_N in [15]. It would be interesting to obtain a direct uniform proof of this equality which is not based on case-by-case calculations. It would also be interesting to understand the meaning of the right hand side of formula (3) with H_1 replaced by H_m.

Remark. The results of this section may be interpreted in the context of the theory of twisted modules over vertex algebras. Consider the lattice vertex algebra V_Q corresponding to the root lattice Q of type X_N, where $X = ADE$. This vertex algebra is isomorphic to the basic representation of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ of type $X_N^{(1)}$ in the homogeneous realization, viewed as a vertex algebra (see, e.g., [11]). Hence σ-twisted modules over V_Q, where σ is an automorphism of $\mathfrak{g}$ preserving the corresponding Cartan subalgebra, realize modules over the twisted affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\sigma$ [2]. In particular, taking the Coxeter transformation M as the automorphism σ, we obtain the basic representation of $\widehat{\mathfrak{g}}_\sigma$ in the principal realization. The corresponding twisted operators are equal to $\Gamma^{\pm\alpha_i}(\zeta)$ up to scalar multiples. The OPE between the twisted vertex operators may be found from the OPE between the corresponding untwisted vertex operators, and this gives a way to compute these scalar multiples (see [2, 13]).

This observation allows us to generalize the results of this paper to the case of more general singularities. In this case the role of an affine Kac–Moody algebra is played by the lattice vertex algebra associated to the Milnor lattice of the singularity (the middle cohomology of the generic fiber of the singularity equipped with the intersection pairing). Using the Picard–Lefschetz periods associated to the singularity, we define twisted vertex operators which realize twisted modules over this lattice vertex algebra.

In the next two sections we describe this in the case of simple singularities, leaving the general case for a subsequent paper. □

3. Vertex operators from singularities

In this section we recall the setup of [7] and describe the integrable hierarchies associated to simple singularities. We will then identify them with the Kac–Wakimoto hierarchies in a uniform way for all ADE types.

3.1. **Periods associated to isolated critical points.** Suppose that we are given a polynomial $f : \mathbb{C}^{2l+1} \to \mathbb{C}$ which has an isolated singularity of multiplicity N at the origin. Denote by H the local algebra $\mathbb{C}[[x_1, \ldots, x_{2l+1}]]/(\partial_{x_1} f, \ldots, \partial_{x_{2l+1}} f)$ of the singularity. We have $\dim H = N$. Let the family $f_t, t \in \mathcal{T} \subset \mathbb{C}^N$, be a miniversal deformation of f, i.e., $f_0 = f$ and $\partial_{t^a} f_t|_{t=0},\ a = 1, \ldots, N$, represent a basis in H. By picking a small ball B_ρ^{2l+1} of dimension $2l+1$ in $\mathbb{C}^{2l+1}$ centered at 0, we can find sufficiently small disk B_δ^1 in $\mathbb{C}$ and ball $\mathcal{T} \subset \mathbb{C}^N$, so that the fibers $f_t^{-1}(\lambda)$, $(\lambda, t) \in B_\delta^1 \times \mathcal{T}$ intersect transversely the boundary of B_ρ^{2l+1}. We may, and will, assume without loss of generality that the critical values of f_t are contained in a disk $B_{\delta_0}^1$ with radius $\delta_0 < 1 < \delta$.

Each tangent space $T_t\mathcal{T}$ is identified with the algebra of functions on the critical scheme $Crit(f_t)$ by the map $\partial_{t^a} \mapsto \partial f/\partial t^a \pmod{\partial f_t/\partial x_i}(1 \le a \le N, 1 \le i \le 2l+1)$. The induced multiplication on $T_t\mathcal{T}$ is denoted by $\bullet_t$. Functions f_t of the miniversal family restricted to their critical schemes $Crit(f_t) \cong T_t\mathcal{T}$ define a vector field on $\mathcal{T}$ denoted by E and called the *Euler field*. Given a volume form ω on B_ρ^{2l+1}, the following *residue pairing* defines a non-degenerate bilinear pairing on $T\mathcal{T}$:

$$(\partial_{t^a}, \partial_{t^b})_t = \Big(\frac{1}{2\pi i}\Big)^{2l+1} \int_{\Gamma_\epsilon} \frac{\partial_{t^a} f_t\ \partial_{t^b} f_t\ \omega}{\frac{\partial f_t}{\partial x_1} \cdots \frac{\partial f_t}{\partial x_{2l+1}}},$$

where the integration cycle Γ_ϵ is supported on $|\frac{\partial f_t}{\partial x_1}| = \ldots = |\frac{\partial f_t}{\partial x_{2l+1}}| = \epsilon$. According to K. Saito's theory of *primitive forms* [14], there exists a volume form ω on B_ρ^{2l+1}, possibly depending on $t \in \mathcal{T}$, such that the residue pairing on $T\mathcal{T}$ is flat homogeneous (of certain degree) with respect to the Euler vector field. Moreover, in the flat coordinate system of the residue metric, the Gauss–Manin connections for various period maps that one can associate to the miniversal deformation of a singularity simultaneously assume a rather simple form. The resulting datum defines on $\mathcal{T}$ a *conformal Frobenius structure* (in the terminology of B. Dubrovin [4]). In the following paragraphs, we introduce relevant notation. For the actual construction of the Frobenius structures in singularity theory we refer the reader to C. Hertling's book [8].

First, assuming that the form ω is fixed once and for all, denote by $(\tau^1, \ldots, \tau^N)$ a coordinate system on $\mathcal{T}$ which is flat with respect to the residue metric, and write ∂_a for ∂_{τ^a}. It follows from the homogeneity condition of the metric that in a suitable flat coordinate system the Euler vector field is the sum of a constant and linear vector fields:

$$E = \sum (1 - d_a)\tau^a \partial_a + \sum \rho_a \partial_a.$$

The constant part represents the class of f in H, and the spectrum of degrees $d_1, \dots, d_N$ ranges from 0 to Δ (called the *conformal dimension* of the Frobenius structure at hand) and differs by a shift from the *Steenbrink spectrum* $\{s_a\}$ of the singularity, $s_a = d_a - 1/2 - \Delta/2 + l$.

Let $V_{\lambda,t} := f_t^{-1}(\lambda) \cap B_\rho^{2l+1}$ denote the Milnor fibers. Choosing $(\lambda, \tau) = (1, 0)$ as a reference point in $B_\delta^1 \times \mathcal{T}$, pick a middle homology class $\varphi \in H_{2l}(V_{(1,0)}; \mathbb{Z}) \cong \mathbb{Z}^N$ in the Milnor lattice, and denote by $\varphi_{\lambda,t}$ its parallel transport (using the Gauss–Manin connection) to the Milnor fiber $V_{\lambda,t}$. Let $d^{-1}\omega$ mean any $2l$-form whose differential is ω. We can integrate $d^{-1}\omega$ over $\varphi_{\lambda,t}$ and obtain this way multivalued functions of λ and t ramified around the discriminant in $B_\delta^1 \times \mathcal{T}$ (over which Milnor fibers become singular). We associate to φ the following Picard–Lefschetz *period vector* $I_\varphi^{(k)}(\lambda, \tau) \in H$ $(k \in \mathbb{Z})$:

$$(I_\varphi^{(k)}(\lambda, \tau), \partial_a) := (2\pi)^{-l} \; (-\partial_a) \; \partial_\lambda^{l+k} \int_{\varphi_{\lambda,\tau}} d^{-1}\omega. \tag{12}$$

Note that this definition is consistent with the operation of stabilization of singularities. Namely, adding the squares of two new variables does not change the RHS since it is offset by an extra differentiation $(2\pi)^{-1}\partial_\lambda$. In particular, this defines the period vector for negative values of k, since we may take $k \geq -l$ in this formula with l as large as needed.

The period vectors (12), being flat sections of a Gauss–Manin connection, satisfy a system of linear differential equations (see [8], section 11)

$$\partial_a I^{(k)} = -(\partial_a \bullet_t)\partial_\lambda I^{(k)}, \; 1 \leq a \leq N, \quad (\lambda - E\bullet_t)\partial_\lambda I^{(k)} = (\Theta - k - 1/2) I^{(k)}, \tag{13}$$

where $\Theta : H \to H$ (called sometimes *Hodge grading operator* of the Frobenius structure) is the operator anti-symmetric with respect to the residue pairing and defined by

$$\Theta(\partial_a) = \theta_a \partial_a, \quad \theta_a = \frac{\Delta}{2} - d_a.$$

Using the last equation in (13) we analytically extend the period vectors to all $|\lambda| > \delta$. It also follows from (13) that the period vectors have the following symmetry:

$$I_\varphi^{(n)}(\lambda, \tau) \;=\; I_\varphi^{(n)}(0, \tau - \lambda \mathbf{1}),$$

where $\tau \mapsto \tau - \lambda\mathbf{1}$ denotes the time-λ translation in the direction of the flat vector field $\mathbf{1}$ obtained from $1 \in H$. (It represents the unit element for all the products $\bullet_t$.)

3.2. **Vertex operators.** Let $\mathcal{H} := H((z^{-1}))$ be the space of formal Laurent series in the indeterminate z^{-1}, equipped with the following symplectic structure:

$$\Omega(\mathbf{f}(z), \mathbf{g}(z)) = \mathrm{Res}_{z=0}(\mathbf{f}(-z), \mathbf{g}(z))dz, \quad \mathbf{f}, \mathbf{g} \in \mathcal{H}.$$

Given a sum $\mathbf{f} = \sum \mathbf{f}_k z^k$, possibly infinite in both directions, we define the vertex operator

$$e^{\widehat{\mathbf{f}}} := \exp\Big(\sum_{k\geq 0}(-1)^{k+1}\sum_{a=1}^N(\mathbf{f}_{-1-k}, [\psi_a]) q_k^a/\sqrt{\hbar}\Big) \exp\Big(-\sum_{k\geq 0}\sum_{a=1}^N(\mathbf{f}_k, [\psi^a])\sqrt{\hbar}\partial_{q_k^a}\Big),$$

where $\{\psi_a\}$ and $\{\psi^a\}$, are dual bases in H, $(\psi^a, \psi_b) = \delta^a_b$. The vertex operator $e^{\widehat{\mathbf{f}}}$ acts on the Fock space of formal series in q^a_k whose coefficients are formal Laurent series in $\sqrt{\hbar}$. Applying this construction to the series

$$\mathbf{f}^{\varphi}_{\tau}(\lambda) := \sum_{k\in\mathbb{Z}} I^{(k)}_{\varphi}(\lambda,\tau)(-z)^k,$$

where $I^{(k)}_{\varphi}$ are the period vectors (12), we obtain vertex operators which will be denoted by $\Gamma^{\varphi}_{\tau}(\lambda)$.

Let $\alpha \in H_{2l}(V_{1,0},\mathbb{Z})$ be a vanishing cycle. Note that the vertex operator $\Gamma^{\alpha}_{\tau}(\lambda)$ depends on the choice of the path connecting $(1,0)$ with (λ,τ). However, changing this path corresponds to acting on α by a monodromy transformation, which gives another vanishing cycle α'. Thus, under this change of path $\Gamma^{\alpha}_{\tau}(\lambda)$ becomes the vertex operator corresponding to the vanishing cycle α', so that the collection $\{\Gamma^{\alpha}_{\tau}(\lambda)\}_{\alpha\in H_{2l}(V_{1,0},\mathbb{Z})}$ is independent of any choices.

Theorem 5. *Let $\alpha,\beta \in H_{2l}(V_{1,0},\mathbb{Z})$ be two vanishing cycles and $\tau\in\mathcal{T}$ be an arbitrary point. The following OPE formula holds:*

$$\Gamma^{\alpha}_{\tau}(\lambda)\Gamma^{\beta}_{\tau}(\mu) = \exp\Big(\int_{\tau'+(\mu-u')\mathbf{1}}^{\tau} I^{(0)}_{\alpha}(\lambda,t)\bullet_t I^{(0)}_{\beta}(\mu,t)\Big) :\Gamma^{\alpha}_{\tau}(\lambda)\Gamma^{\beta}_{\tau}(\mu): \;, \tag{14}$$

where (u',τ') is a generic point in the discriminant.

Here the period $I^{(0)}_{\alpha}(\lambda,t)$ should be expanded in a neighborhood of $\lambda=\infty$. Using the residue pairing, we interpret the integrand in (14) as a formal Laurent power series in λ^{-1} whose coefficients are multivalued 1-forms on $\mathcal{T}$. The integration path C is such that the corresponding path $(\mu,t), t\in C$, does not intersect the discriminant, and the cycle $\beta\in H_{2l}(V_{\mu,\tau},\mathbb{Z})$ vanishes when transported to $H_{2l}(V_{u',\tau'},\mathbb{Z})$.

Proof. It follows from the definition of normal ordering that in order to prove (14) it is enough to show that the integral in the exponent equals $\Omega(\mathbf{f}^{\alpha}_{\tau}(\lambda)_+,\mathbf{f}^{\beta}_{\tau}(\mu)_-)$, i.e., we need to show that this expression vanishes for $\tau=\tau'+(\mu-u')\mathbf{1}$ and that its differential equals the integrand in (14). The first of these two statements is obvious because $I^{(-n-1)}_{\beta}(\mu,\tau')=0$ for $n\geq 0$ and $\mu=u'$. For the differential we have

$$d\Omega(\mathbf{f}^{\alpha}_{\tau}(\lambda)_+,\mathbf{f}^{\beta}_{\tau}(\mu)_-) = \sum_{n\geq 0}\sum_{a=1}^{N}(-1)^{n+1}\partial_a\left(I^{(n)}_{\alpha}(\lambda,\tau), I^{(-n-1)}_{\beta}(\mu,\tau)\right)\, d\tau^a.$$

Using the Leibniz rule and the differential equations (13) we get

$$\sum_{n\geq 0}\sum_{a=1}^{N}(-1)^n\Big(\Big((\partial_a\bullet)I^{(n+1)}_{\alpha}, I^{(-n-1)}_{\beta}\Big)+\Big(I^{(n)}_{\alpha},(\partial_a\bullet)I^{(-n)}_{\beta}\Big)\Big)d\tau^a.$$

Applying the Frobenius property of the multiplication $\bullet$, we find that this is equal to

$$=\sum_{n\geq 0}\sum_{a=1}^{N}(-1)^n\Big(\Big(I^{(n+1)}_{\alpha}\bullet I^{(-n-1)}_{\beta},\partial_a\Big)+\Big(I^{(n)}_{\alpha}\bullet I^{(-n)}_{\beta},\partial_a\Big)\Big)d\tau^a.$$

Finally, in the above sum, with respect to the summation over n, all terms cancel except for $\left(I^{(0)}_\alpha \bullet I^{(0)}_\beta, \partial_a\right) d\tau^a$. Formula (14) follows. $\square$

3.3. **Simple singularities and the corresponding Hirota equations.** Consider the case of a simple singularity f of type X_N, where $X = ADE$. In this case the Milnor lattice (the middle homology $H_{2l}(V_{1,0}, \mathbb{Z})$ of the reference Milnor fiber $V_{1,0}$) is identified with the root lattice of the corresponding simple Lie algebra $\mathfrak{g}$ of type X_N. The invariant inner product $(\cdot\,|\,\cdot)$ on the root lattice is identified up to the sign $(-1)^l$ with the intersection pairing on the Milnor lattice. The monodromy group acts on the Milnor lattice as the Weyl group. The roots are identified with vanishing cycles, and the monodromy group action on them is generated by reflections $\varphi \mapsto \varphi - \langle \varphi, \alpha\rangle \alpha$ in the vanishing cycles. The *classical monodromy operator* is by definition the parallel transport of the cycles along the loop $\theta \mapsto (\lambda, \tau) = (\exp i\theta, 0)$ and hence coincides with a Coxeter element M of the root system. We keep the notation α_i of Section 2 for the representatives in the orbits of M in the set of Nh vanishing cycles.

In [7], a Hirota bilinear equation was associated to a simple singularity of type ADE in the following form:

$$\operatorname{Res}_{\lambda=\infty} \frac{d\lambda}{\lambda} \sum_\alpha \widetilde{a}_i\ \Gamma^\alpha_0(\lambda) \otimes \Gamma^{-\alpha}_0(\lambda)\ \tau\otimes\tau = \frac{N(h+1)}{12}\ \tau\otimes\tau + \tag{15}$$
$$\sum_{k=0}^\infty \sum_{a=1}^N (m_a + kh)(q^a_k \otimes 1 - 1 \otimes q^a_k)(\partial_{q^a_k} \otimes 1 - 1 \otimes \partial_{q^a_k})\ \tau\otimes\tau.$$

The sum here is taken over all roots, but the coefficients $\widetilde{a}_i$ are the same for all roots from the M-orbit of α_i. They are defined by the formula

$$\widetilde{a}_i := h \lim_{\epsilon\to 0} \exp\Big(- \int_{-\mathbf{1}}^{\tau_0 - \epsilon\mathbf{1}} I^{(0)}_{\alpha_i}(0,t) \bullet I^{(0)}_{\alpha_i}(0,t) - \int_1^\epsilon \frac{2d\xi}{\xi} - 4\ln 2\Big). \tag{16}$$

The integration here should be understood as follows. We pick a generic point $\tau_0 := \tau' - u'\mathbf{1}$ on the discriminant. Since α_i is a vanishing cycle, there exists a path from $(1,0)$ to (τ', u') such that α_i vanishes when transported along it to the end point. The integration is performed along such a path. For more details, see [7].

It is straightforward to check (see [7], Section 8) that the vertex operators $\Gamma^{\alpha_i}(\zeta)$ of representation theory given by formula (2) turn into the vertex operators $\Gamma^{\alpha_i}_0(\lambda)$ of singularity theory after the the following rescaling of the dynamical variables:

$$q^a_k = \sqrt{\hbar}\ \prod_{r=0}^k (m_a + rh) y_{m_a + kh} \tag{17}$$

and substitution $\lambda = \zeta^h/h$.

Note that

$$\lim_{\zeta\to w} (1 - w^h/\zeta^h)/(1 - w/\zeta) = h.$$

Using this, we find from Theorem 5 (applied at $\tau = 0$) and formulas (2) and (8) that

$$a_i = h \lim_{\lambda\to\mu} (1-\mu/\lambda)^{-2} \exp\left(- \int_0^{\tau_0 + \mu\mathbf{1}} I^{(0)}_{\alpha_i}(\lambda,t) \bullet I^{(0)}_{\alpha_i}(\mu,t)\right). \tag{18}$$

Taking into account that $\operatorname{Res} d\lambda/\lambda = \operatorname{Res} d\zeta/\zeta$, in order to prove that $a_i = \widetilde{a}_i$ (which is the statement of the Theorem in the Introduction) it remains to identify this limit with the one in (16).

Lemma 6. *We have $a_i = \widetilde{a}_i$ for all $i = 1, \ldots, N$.*

Proof. Consider the family of integrals

$$\int_{-\mathbf{1}}^{\tau_0 - \epsilon\mathbf{1}} I^{(0)}_{\alpha_i}(\lambda - \mu, t) \bullet I^{(0)}_{\alpha_i}(0, t) + \int_1^{\lambda-\mu+\epsilon} \frac{2d\xi}{\xi}. \tag{19}$$

Here τ_0 lies on the discriminant and the first integral is computed along a discriminant-avoiding path connecting the reference point $-\mathbf{1}$ with a neighborhood of τ_0 in such a way that the cycle α_i, transported along this path, vanishes as $\epsilon \to 0$. When $\lambda - \mu = 0$, the integrals coincide with those in the exponent of (16), and altogether tend to $-\ln(\widetilde{a}_i/h) - 4\ln 2$ (up to an integer multiple of $2\pi i$) in the limit $\epsilon \to 0$.

Now set $\epsilon = 0$ in (19), and write the first integral as the sum $\int_{-\mathbf{1}}^{-\mu\mathbf{1}} + \int_{-\mu\mathbf{1}}^{\tau_0}$. The second integral may be converted into the one in the exponent of (18) since the integrand is invariant under the shifts $(\lambda, \mu, t) \mapsto (\lambda + c, \mu + c, t + c\mathbf{1})$. The first integral has a well-defined limit as $\mu \to \lambda$, which is equal to $-\int_1^\lambda 2d\xi/\xi$. Indeed, in this limit, and when $t = \xi\mathbf{1}$, we have (see [7])

$$I^{(0)}_{\alpha_i}(0, \xi\mathbf{1}) \bullet I^{(0)}_{\alpha_i}(0, \xi\mathbf{1}) = -(\alpha_i \,|\, \alpha_i)\rangle \frac{d\xi}{\xi} = -2\frac{d\xi}{\xi}. \tag{20}$$

Note that

$$\exp\left(\int_1^{\lambda-\mu} \frac{2d\xi}{\xi} - \int_1^{\lambda} \frac{2d\xi}{\xi}\right) = (1 - \mu/\lambda)^2.$$

Thus, at $\epsilon = 0$, the integral (19) tends in the limit $\lambda - \mu \to 0$ to $-\ln(a_i/h)$.

Now write the first integral in (19) as the sum $\int_{-\mathbf{1}}^{\tau} + \int_{\tau}^{\tau_0 - \epsilon\mathbf{1}}$, where τ is any fixed point along the path. The first summand depends continuously on λ, μ and ϵ up to $\lambda = \mu = \epsilon = 0$. Thus, in determining if the limits of (19) as $\epsilon \to 0$ and $\lambda - \mu \to 0$ commute, we can replace the base point $-\mathbf{1}$ by τ, which can be chosen to lie in a neighborhood of the discriminant point τ_0.

Components of the period vector $I^{(0)}_{\alpha_i}(\lambda, t)$ near a non-degenerate critical point where the cycle α_i vanishes are proportional to

$$\frac{1}{\sqrt{\lambda - u}} \left(1 + O(\lambda - u)\right),$$

where u is the critical value, considered as a function of t and taken equal 0 at $t = \tau_0$. Thus, the integrand $I^{(0)}_{\alpha_i}(\lambda - \mu, t) \bullet I^{(0)}_{\alpha_i}(0, t)$ with small $\lambda - \mu$ has three types of singularities near $t = \tau_0$:

$$\frac{\sqrt{\lambda - \mu - u}}{\sqrt{-u}}\, du, \quad \frac{\sqrt{-u}\; du}{\sqrt{\lambda - \mu - u}}, \quad \text{and} \quad \frac{du}{\sqrt{\lambda - \mu - u}\sqrt{-u}}. \tag{16}$$

The first singularity is integrable, which makes the order of passing to the limit irrelevant (indeed, $\int_{-\epsilon}^0$ tends to 0 as $\epsilon \to 0$, uniformly in $\lambda - \mu$).

The same is true for the second singularity in (16), which reduces to the first one by integration by parts.

The remaining case literally coincides with the integral (19) for the A_1 singularity $x_1^2/2+x_2x_3+u=\lambda$. In this case the result may be derived from [7]. However, for the sake of completeness we give a direct proof here.

In the A_1 case, we have

$$I_\alpha^{(0)}(\lambda,u)=\frac{\pm 2}{\sqrt{2(\lambda-u)}}.$$

Hence (19) turns into

$$\int_{-1}^{-\epsilon}\frac{4\ du}{\sqrt{2(\lambda-\mu-u)}\sqrt{2(-u)}}+\int_1^{\lambda-\mu+\epsilon}\frac{2d\xi}{\xi}. \tag{17}$$

When $\lambda-\mu=0$, the integrals cancel (modulo $2\pi i\mathbb{Z}$). On the other hand, when $\epsilon=0$, (17) becomes

$$-2\ln\left(-u+\frac{\lambda-\mu}{2}+\sqrt{(\lambda-\mu-u)(-u)}\right)\Bigg|_{-1}^{0}+2\ln\xi\ \Bigg|_1^{\lambda-\mu}=$$
$$-2\ln\frac{\lambda-\mu}{2}+2\ln\left(1+\frac{\lambda-\mu}{2}+\sqrt{1+\lambda-\mu}\right)+2\ln(\lambda-\mu).$$

In the limit $\lambda-\mu\to 0$ this tends to $4\ln 2$ as desired. This completes the proof. $\square$

This implies the main result of this paper (see the Theorem in the Introduction): The hierarchy associated in [7] to a simple singularity of type X_N, where $X=ADE$, coincides with the Kac–Wakimoto hierarchy for the Lie algebra $X_N^{(1)}$.

4. Period realizations of the basic representation

In the previous section we made use of the vertex operators Γ_τ^α defined in terms of the Picard–Lefschetz periods of a singularity. In this section we will show in the case of simple singularities how these operators may be used to construct realizations of the basic representation of the corresponding affine Kac–Moody algebra. In the special case $\tau=0$ of the unperturbed singularity, the realization coincides, up to the change of variables (17), with the principal realization we use in Section 2. Hence we obtain a family of deformations of this principal realization. It is an interesting question whether this family has a representation theoretic interpretation.

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}^M(\cong\widehat{\mathfrak{g}})$ of type X_N is spanned by K, d, $\varphi_{\pm m},\ m\in E_+$, and $A_{\alpha,m},\ m\in\mathbb{Z}$, where

$$\varphi_m=\frac{1}{h}\sum_{k=1}^{h}M^k(\varphi t^m),\quad A_{\alpha,m}=\frac{1}{h}\sum_{k=1}^{h}M^k(A_\alpha t^m).$$

Here $A_\alpha\in\mathfrak{g}_\alpha$ (α is a root), and φ is any element of the Cartan subalgebra $\mathfrak{h}$. We continue to identify the Coxeter transformation $M:\mathfrak{h}\to\mathfrak{h}$ with the operator of classical monodromy on the homology group $H_{2l}(V_{1,0},\mathbb{C})$, which in its turn is identified by

$\varphi \mapsto I^{(0)}_\varphi(1,0)$ (a period map composed with the residue pairing operator) with the local algebra H of the singularity. We form generating functions

$$\varphi(\zeta) := \sum_{m\in E_+} \varphi_m \frac{\zeta^{-m}}{-m} + \varphi_{-m}\frac{\zeta^m}{m}, \quad x_\alpha(\zeta) := \sum_{m\in\mathbb{Z}} A_{\alpha,m}\zeta^{-m}.$$

For each $(\lambda,\tau)\in\mathbb{C}\times\mathcal{T}$, introduce an element of the Heisenberg algebra $H[[z,z^{-1}]]$

$$\mathbf{f}^\varphi_\tau(\lambda) := \sum_{n\in\mathbb{Z}} I^{(n)}_\varphi(\lambda,\tau)(-z)^n.$$

To specify the branch of the multivalued period vectors $I^{(n)}_\varphi(\lambda,\tau)$, we fix a path avoiding the discriminant and such that it connects (λ,τ) with $(1,0)$ and transports the cycle φ along it. As functions of λ near $\lambda=\infty$, the periods expand in Laurent series in $\lambda^{1/h}$. In what follows we put

$$\zeta = (h\lambda)^{1/h}.$$

Recall that our vertex operators are defined as quantizations of the Heisenberg group elements $\Gamma^\alpha_\tau(\lambda) = :\!\exp(\widehat{\mathbf{f}}^\alpha_\tau(\lambda)\!:$ (as in Section 3.2), where α is any root.

Introduce *phase factors*

$$U_\alpha(\lambda,\tau) = \exp\Big(\frac{1}{2}\int_{-\mathbf{1}}^{\tau-\lambda\mathbf{1}} I^{(0)}_\alpha(0,t)\bullet_t I^{(0)}_\alpha(0,t)\Big), \quad \tau\in\mathcal{T}. \tag{21}$$

The integrand is a covector on $\mathcal{T}$ applied at the point t and obtained, using the residue pairing identification $T^*_t\mathcal{T}\cong T_t\mathcal{T}$, from the product $\bullet_t$ of two (equal) period vectors. The 1-form consisting of these covectors is integrated along a path (between the point $\tau-\lambda\mathbf{1}$ and the point $-\mathbf{1}$).

Finally, introduce the *energy operator*

$$l_\tau = z\partial_z + \frac{1}{2} - \Theta + z^{-1}E\bullet_\tau \;\; : \mathcal{H}\to\mathcal{H},$$

where E is the Euler vector field, and Θ is the Hodge grading operator (see Section 3.1). The energy operator acts on the symplectic loop space $\mathcal{H}=H((1/z))$ by infinitesimal symplectic transformation, and so its quantization $\hat{l}_\tau$ acts on the Fock space. We recall that for any infinitesimal symplectic transformation, A, one expresses its quadratic Hamiltonian $\frac{1}{2}\Omega(A\mathbf{f},\mathbf{f})$ in Darboux coordinates $\{q^a_k, p_{l,b}\,|\, a,b=1,\dots,N,\ k=0,1,2,\dots\}$ associated with the polarization $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_- = T^*\mathcal{H}_+$ and then defines $\widehat{A}$ in the standard way:

$$(q^a_k q^b_l)\widehat{\ } = q^a_k q^b_l/\hbar, \quad (q^a_k p_{l,b})\widehat{\ } = (p_{l,b}q^a_k)\widehat{\ } = q^a_k\frac{\partial}{\partial q^b_l}, \quad (p_{k,a}p_{l,b})\widehat{\ } = \hbar\frac{\partial^2}{\partial q^a_k\partial q^b_l}. \tag{22}$$

Theorem 7. *There exist constants c_α such that for each τ the formulas:*

$$\begin{aligned}
&\varphi(\zeta) \;\mapsto \widehat{\mathbf{f}}^\varphi_\tau(\lambda),\\
&x_\alpha(\zeta) \mapsto c_\alpha\ \lambda^{(\alpha|\alpha)/2}\ U_\alpha(\lambda,\tau)\ \Gamma^\alpha_\tau(\lambda),\\
&K\mapsto 1/h, \quad d\mapsto \widehat{l}_\tau
\end{aligned}$$

define on the Fock space a representation of the affine Kac–Moody algebra equivalent to the basic representation of level 1.

Proof. When $\tau = 0$, the formulas coincide (up to the rescaling (17)) with the principal realization of the basic representation of level 1 [9, 10, 12].

To derive the theorem for an arbitrary τ, we introduce intertwining operators $\widehat{S}_\tau$ as quantizations of a certain symplectic transformations

$$S_\tau(z) = 1 + S_\tau^{(1)} z^{-1} + S_\tau^{(2)} z^{-2} + \cdots, \quad S_\tau^{(k)} \in \mathrm{End}(H), \quad S_\tau^*(-z) S_\tau(z) = 1. \tag{23}$$

By definition, $\widehat{S}_\tau = \exp^{\widehat{\ln S_\tau}}$.

The series S_τ (also known as *calibration* of the corresponding Frobenius structure) is defined as follows. We introduce one more period vector, $J(\tau, z)$, corresponding to the *complex oscillating integrals*

$$(J(\tau, z), \partial_a) = (2\pi z)^{-l-\frac{1}{2}} \ (z\partial_a) \int e^{f_\tau(x)/z} \omega, \quad 1 \le a \le N.$$

The cycles of integration here are "Lefschetz' timbles," i.e., relative homology classes of $\mathbb{C}^{2l+1}$ modulo $(\mathrm{Re}(f_\tau/z))^{-1}[-R, -\infty)$ in the limit $R \to \infty$.

When ω is a primitive volume form, the oscillating integrals (which are related to the periods $I_\varphi^{(k)}(\lambda, \tau)$ by Laplace-like transforms in λ, see for instance [7]) satisfy the following system of differential equations:

$$z\partial_a J(\tau, z) = (\partial_a \bullet_\tau) J(\tau, z), \ 1 \le a \le N, \quad (z\partial_z + E)J = \Theta J,$$

where Θ is the Hodge grading operator. Thus, J is a fundamental solution of a flat connection on the vector bundle with the fiber H over $(z, \tau) \in (\mathbb{C} - 0) \times \mathcal{T}$. The second equation, which expresses homogeneity properties of oscillating integrals (where $\deg z = 1$), may be rewritten as

$$\nabla_\tau J = 0, \quad \text{where} \quad \nabla_\tau = \partial_z + z^{-2} E \bullet_\tau - z^{-1}\Theta.$$

One may think of ∇_τ as an *isomonodromic* family of connection operators over $\mathbb{C} - 0$, parametrized by $\tau \in \mathcal{T}$. The operators S_τ are defined as gauge transformations of the form (23) conjugating ∇_τ and $\nabla_0 = \partial_z - z^{-1}\Theta$:

$$\nabla_\tau = S_\tau \nabla_0 S_\tau^{-1}.$$

In the ADE case, one can show that S_τ satisfying the initial condition $S_0 = 1$ exists and is unique. It follows that

$$l_\tau = S_\tau \, l_0 \, S_\tau^{-1}, \quad \text{and} \quad z\partial_a S_\tau = (\partial_a \bullet_\tau) S_\tau, \ 1 \le a \le N.$$

Note that the quantization (22) defines a representation of the Poisson algebra consisting of quadratic Hamiltonians involving only pq and q^2 terms. Both $\widehat{\ln S_\tau}$ and $\widehat{l_\tau}$ are obtained by quantizing such Hamiltonians. Therefore, $\widehat{l_\tau} = \widehat{S}_\tau \widehat{l}_0 \widehat{S}_\tau^{-1}$.

Furthermore, $S_\tau \mathbf{f}_0^\varphi = \mathbf{f}_\tau^\varphi$. Indeed, both sides satisfy the same differential equation with respect to τ and the same initial condition at $\tau = 0$. Therefore the vertex operators Γ_τ^α and Γ_0^α, being elements of the Heisenberg group corresponding to the elements $\mathbf{f}_\tau$

and $\mathbf{f}_0$ of the Heisenberg algebra, are conjugated by $\widehat{S}_\tau$ up to certain scalar factors. The precise values of the factors have been found in [7], Section 5, Theorem A:

$$U_\alpha(\lambda,\tau)\Gamma_\tau^\alpha(\lambda) = \widehat{S}_\tau\ U_\alpha(\lambda,0)\Gamma_0^\alpha(\lambda)\ \widehat{S}_\tau^{-1}.$$

Thus, $\widehat{S}_\tau^{-1}$ intertwines the operators described in the theorem with the operators defining the level 1 basic representation of the affine Kac–Moody algebra. □

References

[1] V. Arnold, S. Gusein-Zade, and A. Varchenko, *Singularities of Differentiable maps.* Vol. II. Monodromy and Asymptotics of Integrals. Birkhäuser Boston, 1988.

[2] B. Bakalov and V. Kac, *Twisted modules over lattice vertex algebras*, in Lie theory and its applications in physics V, eds. H.-D. Doebner and V.K. Dobrev, pp. 3–26, World Scientific Publishing, 2004.

[3] R. Borcherds, *Vertex algebras, Kac–Moody algebras, and the Monster,* Proc. Natl. Acad. Sci. USA **83** (1986) 3068–3071.

[4] B. Dubrovin, *Geometry of 2D topological field theories,* in Integrable Systems and Quantum Groups, Lecture Notes in Mathematics, Vol. 1620, Springer-Verlag, New York, 1996, 120–348.

[5] I. Frenkel and V. Kac, *Basic representation of affine Lie algebras and dual resonance models,* Invent. Math. **62** (1980) 23–66.

[6] A. Givental, A_{n-1} *singularities and* n*KdV hierarchies,* Mosc. Math. J. **3** (2003) 475–505, 743.

[7] A. Givental and T. Milanov, *Simple singularities and integrable hierarchies*, in The breadth of symplectic and Poisson geometry, pp. 173–201, Progr. Math. **232,** Birkhäuser, 2005.

[8] C. Hertling, *Frobenius Manifolds and Moduli Spaces for Singularities.* Cambridge University Press, Cambridge, UK, 2002.

[9] V.G. Kac, D.A. Kazhdan, J. Lepowsky and R.L. Wilson, *Realization of the basic representations of the Euclidean Lie algebras*, Adv. Math. **42** (1981) 83–112.

[10] V. Kac, *Infinite-dimensional Lie algebras.* Third Edition. Cambridge University Press, 1990.

[11] V. Kac, *Vertex algebras for beginners*, Second Edition. University Lecture Series **10**, AMS, 1998.

[12] V. Kac and M. Wakimoto, *Exceptional hierarchies of Soliton Equations.* Proc. Symp. Pure Math. **49**, pp. 138–177, AMS, 1989.

[13] J. Lepowsky, *Calculus of twisted vertex operators*, Proc. Natl. Acad. Sci. USA **82** (1985) 8295–8299.

[14] K. Saito, *On periods of primitive integras, I.* Preprint RIMS, 1982.

[15] C.-Z. Wu, *A Remark on Kac-Wakimoto Hierarchies of D-type*, Preprint arXiv:0906.5360.

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