GROMOV – WITTEN INVARIANTS AND QUANTIZATION OF QUADRATIC HAMILTONIANS

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ABSTRACT. We describe a formalism based on quantization of quadratic hamiltonians and symplectic actions of loop groups which provides a convenient home for most of known general results and conjectures about Gromov-Witten invariants of compact symplectic manifolds and, more generally, Frobenius structures at higher genus. We state several results illustrating the formalism and its use. In particular, we establish Virasoro constraints for semisimple Frobenius structures and outline a proof of the Virasoro conjecture for Gromov – Witten invariants of complex projective spaces and other Fano toric manifolds. Details will be published elsewhere.

1. Gromov – Witten invariants.

Let X be a compact almost Kähler manifold of complex dimension D. Denote by $X_{g,m,d}$ the moduli (orbi)space of degree d stable holomorphic maps to X of genus g curves with m marked points [27, 3]. The degree d takes values in the lattice $H_2(X)$. The moduli space is compact and can be equipped [2, 29, 36] with a rational coefficient *virtual fundamental cycle* $[X_{g,m,d}]$ of complex dimension $m + (1 - g)(D - 3) + \int_d c_1(TX)$.

The *total descendent potential* of X is defined as

$$\mathcal{D}_X := \exp \sum \hbar^{g-1} \mathcal{F}_X^g,$$

where $\mathcal{F}_X^g$ is the *genus g descendent potential*

$$\sum_{m,d} \frac{Q^d}{m!} \int_{[X_{g,m,d}]} \wedge_{i=1}^m (\sum_{k=0}^{\infty} (\text{ev}_i^* t_k) \psi_i^k).$$

Here ψ_i^k are the powers of the 1-st Chern class of the universal cotangent line bundle over $X_{g,m,d}$ corresponding to the i-th marked point, $\text{ev}_i^* t_k$ are pull-backs by the evaluation map $\text{ev}_i : X_{g,m,d} \to X$ at the i-th marked point of the cohomology classes $t_0, t_1, ... \in H^*(X, \mathbf{Q})$, and Q^d is the representative of d in the semigroup ring of the semigroup of degrees of holomorphic curves in X. The *genus g Gromov – Witten potential* of X is defined as the restriction $F_X^g(t) := \mathcal{F}_X^g|_{t_0=t, \ t_1=t_2=...=0}$.

The genus g descendent potentials are considered as formal functions on the super-space of vector Laurent polynomials $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + ...$ in one indeterminate z with coefficients in $H := H^*(X; \mathbf{Q}[[Q]])$, the cohomology space of X over the *Novikov ring*. The latter is a suitable power series completion of the semigroup ring of degrees.

1.1. *Remark.* We will assume further on that H has no odd part and leave the super-space generalization of the material of sections 2 – 5 to the reader. It is not clear at the moment if the content of sections 6 – 10 admits such a generalization.

1.2. *Example.* When $X = point$, the moduli spaces of stable maps coincide with the Deligne – Mumford compactifications $\overline{\mathcal{M}}_{g,m}$ of moduli spaces of genus g Riemann surfaces with m marked points. According to Witten's conjecture [38] proved by Kontsevich [26], the total descendent potential in this case coincides with the tau-function of the KdV-hierarchy satisfying the string equation [25]. We will denote $\mathcal{D}_{point}$ by $\tau(\hbar; \mathbf{t})$ and call it the *Witten – Kontsevich tau-function.*

Taylor coefficients of the Gromov – Witten and descendent potentials depend only on the deformation class of the symplectic structure and are often referred to respectively as Gromov – Witten invariants and their gravitational descendents.

2. Quadratic hamiltonians and quantization.

Let $H, (\cdot, \cdot)$ be an N-dimensional vector space equipped with a non-degenerate symmetric bilinear form. Let $\mathcal{H}$ be the space of Laurent polynomials in one indeterminate z with vector coefficients from H. We introduce a symplectic bilinear form in $\mathcal{H}$ by

$$\Omega(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z)) dz = -\Omega(g, f).$$

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ correspond to the decomposition $f(z, z^{-1}) = f_+(z) + f_-(1/z)/z$ of the Laurent polynomials into polynomial and polar parts. The subspaces $\mathcal{H}_\pm$ are Lagrangian. In particular the projection $\mathcal{H} \to \mathcal{H}_+$ along $\mathcal{H}_-$ defines a polarization of $(\mathcal{H}, \Omega)$. We quantize infinitesimal symplectic transformations L on $\mathcal{H}$ to order ≤ 2 linear differential operators $\hat{L}$. In a Darboux coordinate system $\{p_\alpha, q_\beta\}$ compatible with our decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ we have

$$(p_\alpha p_\beta)\hat{} = \hbar \partial_{q_\alpha} \partial_{q_\beta}, \ (p_\alpha q_\beta)\hat{} = q_\beta \partial_{q_\alpha}, \ (q_\alpha q_\beta)\hat{} = q_\alpha q_\beta / \hbar.$$

Note that $[\hat{F}, \hat{G}] = \{F, G\}\hat{} + \mathcal{C}(F, G)$ where the cocycle $\mathcal{C}$ satisfies

$$\mathcal{C}(p_\alpha^2, q_\alpha^2) = 2, \ \ \mathcal{C}(p_\alpha p_\beta, q_\alpha q_\beta) = 1 \text{ for } \alpha \neq \beta,$$

and $\mathcal{C} = 0$ for any other pairs of quadratic Darboux monomials.

2.1. *Remark: Fock spaces.* The differential operators $\hat{L}$ act on functions of $\hbar$ and $\mathbf{q} = q_0 + q_1 z + q_2 z^2 + \dots$ where $q_0, q_1, q_2, \dots \in H$. We will often refer to such functions as elements of the *Fock space.* However we will be concerned with various functions which belong to different formal series completions of the polynomial Fock space, and we will not describe the completions explicitly in this paper.

2.2. *Dilaton shift.* The total descendent potential $\mathcal{D}_X$ will be considered as a vector in the Fock space corresponding to the space $H = H^*(X; \mathbf{Q}[[Q]])$ equipped with the Poincaré intersection form $(\cdot, \cdot)$. More precisely, we introduce the *dilaton shift* $\mathbf{q}(z) = \mathbf{t}(z) - z$ which identifies the indeterminates $t_0, t_1, \dots$ in the descendent potential with coordinates $q_0, q_1, \dots$ in $\mathcal{H}_+$. Slightly abusing notation we will always identify functions of $\mathbf{t}$ with the functions of $\mathbf{q} \in \mathcal{H}_+$ obtained from them by the dilaton shift. In particular, the genus g potentials $\mathcal{F}_X^g$ are considered as formal functions of $\mathbf{t}$ near the origin $(0, 0, 0, \dots)$ or, equivalently, as formal functions of $\mathbf{q}$ near $(0, -1, 0, \dots)$. The same convention applies to the Witten – Kontsevich tau-function $\tau = \mathcal{D}_{point}$.

3. The Witten – Kontsevich tau-function.

Denote $D = z(d/dz)z$ and put $L_m = z^{-1/2}D^{m+1}z^{-1/2}$ so that

$$L_{-1} = 1/z, \ L_0 = z\frac{d}{dz} + 1/2, \ L_1 = z^3\frac{d^2}{dz^2} + 3z^2\frac{d}{dz} + \frac{3}{4}z,$$

$$L_2 = z^5\frac{d^3}{dz^3} + \frac{15}{2}z^4\frac{d^2}{dz^2} + \frac{45}{4}z^3\frac{d}{dz} + \frac{15}{8}z^2, \ ...$$

It is easy to check that $\Omega(Df, g) = \Omega(f, Dg)$ and that $\Omega(z^{-1/2}f, g) = -i\Omega(f, z^{-1/2}g)$ (whatever it means). This implies $\Omega(L_m f, g) = -\Omega(f, L_m g)$ and shows that the operators L_m are infinitesimal symplectic transformations on $\mathcal{H}$. On the other hand, D is conjugate to $z^2 d/dz = -d/dw$ where $w = 1/z$, and hence L_m commute as $-wd^{m+1}/dw^{m+1}$ and therefore (via the Fourier transform) — as the vector fields $-x^{m+1}d/dx$ on the line. Thus the Poisson brackets satisfy $\{L_m, L_n\} = (m - n)L_{m+n}$, and we have a representation of the Lie algebra of vector fields on the line to the Lie algebra of quadratic hamiltonians on $\mathcal{H}$.

In the case of 1-dimensional H with the standard inner product, using the Darboux coordinate system $f = ... + p_1/z^2 - p_0/z + q_0 + q_1 z + ...$ on $\mathcal{H}$, we get (here $\partial_k = \partial/\partial q_k$):

$$\hat{L}_{-1} = q_0^2/2\hbar + \textstyle\sum_{m\geq 0} q_{m+1}\partial_m$$
$$\hat{L}_0 = \textstyle\sum_{m\geq 0}(m + 1/2)q_m\partial_m$$
$$\hat{L}_1 = \hbar\partial_0^2/8 + \textstyle\sum_{m\geq 0}(m + 1/2)(m + 3/2)q_m\partial_{m+1}$$
$$\hat{L}_2 = 3\hbar\partial_0\partial_1/4 + \textstyle\sum_{m\geq 0}(m + 1/2)(m + 3/2)(m + 5/2)q_m\partial_{m+2}.$$

We have $[\hat{L}_m, \hat{L}_n] = (m - n)\hat{L}_{m+n}$ unless $m, n = \pm 1$, in which case: $[\hat{L}_1, \hat{L}_{-1}] = 2[\hat{L}_0 + 1/16]$. Thus the operators $\hat{L}_m + \delta_{m,0}/16$ form a representation of the Lie algebra of vector fields on the line.

Due to [25], the following result is a reformulation of the Kontsevich theorem [26] confirming the Witten conjecture [38].

3.1. Theorem. *The Witten – Kontsevich tau-function is annihilated by the operators $\hat{L}_m + \delta_{m,0}/16$, $m = -1, 0, 1, 2, ...$, and is completely characterized by this property (up to a scalar factor).*

4. Hodge integrals.

Let $\mathbb{E}$ denote the *Hodge bundle* over the moduli space $X_{g,m,d}$. By definition the fiber of $\mathbb{E}$ over the point represented by a stable map $\Sigma \to X$ is the complex space of dimension g dual to $H^1(\Sigma, \mathcal{O}_\Sigma)$. In fact $\mathbb{E}$ is the pull-back of by the *contraction map* ct $: X_{g,m,d} \to \overline{\mathcal{M}}_{g,m}$ of the Hodge bundle over the Deligne – Mumford space. It is known [32] that even components $ch_{2k}(\mathbb{E})$ of the Chern character vanish. We define the *total Hodge potential* of X as an extension of the total descendent potential depending on the sequence $\mathbf{s} = (s_1, s_2, ...)$ of new variables and incorporating intersection indices with characteristic classes of the Hodge bundles (as in [14]):

$$\mathcal{E}_X(\hbar; \mathbf{t}; \mathbf{s}) := \exp\sum_g \hbar^{g-1}\sum_{m,d}\frac{Q^d}{m!}\int_{[X_{g,m,d}]} e^{\sum_{k=1}^{\infty} s_k \ ch_{2k-1}(\mathbb{E})} \wedge_{i=1}^m \Big(\sum_k (\mathrm{ev}_i^* t_k)\psi_i^k\Big).$$

We consider $\mathcal{E}_X$, subject to the dilaton shift, as a family of elements in the Fock space depending on the parameters $\mathbf{s}$.

On the other hand, it is obvious that multiplication by z^{2k-1} defines an infinitesimal symplectic transformation on $(\mathcal{H}, \Omega)$, and we denote by $(z^{2k-1})^\wedge$ the corresponding quantization.

4.1. Theorem.
$$\mathcal{E}_X = \exp[\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \ s_k \ (z^{2k-1})^\wedge] \ \mathcal{D}_X$$

where B_{2k} are Bernoulli numbers: $x/(1 - e^{-x}) = 1 + x/2 + \sum_{k=1}^{\infty} B_{2k} x^{2k}/(2k)!$.

4.2. *Remark.* The theorem is a reformulation of a result by Faber – Pandharipande [14] which adjusts to arbitrary target spaces the famous Mumford's Riemann – Roch – Grothendieck formula [32] expressing $ch(\mathbb{E})$ via ψ_i.

5. Gravitational ancestors.

Consider the composition $\pi : X_{g,m+l,d} \to \overline{\mathcal{M}}_{g,m+l} \to \overline{\mathcal{M}}_{g,m}$ of the contraction map with the operation of forgetting all marked points except the first m. Let $\bar{\psi}_i := \pi^*(\psi_i)$ denote pull-backs of the classes ψ_i, $i = 1, ..., m$, from $\overline{\mathcal{M}}_{g,m}$. We introduce the *total ancestor potential*

$$\mathcal{A}_t(\hbar; \mathbf{t}) := \exp \sum_{g=0}^{\infty} \hbar^{g-1} \bar{\mathcal{F}}_t^g,$$

where the *genus g ancestor potential* $\bar{\mathcal{F}}_t^g$ is defined by

$$\bar{\mathcal{F}}_t^g := \sum_{m,l,d} \frac{Q^d}{m!l!} \int_{[X_{g,m+l,d}]} \wedge_{i=1}^{m} (\sum_k (\text{ev}_i^* t_k) \bar{\psi}_i^k) \ \wedge_{i=m+1}^{m+l} \text{ev}_i^* t.$$

By definition the sum does not contain the terms with $(g, m) = (0, 0), (0, 1), (0, 2)$ and $(1, 0)$. We treat $\mathcal{A}_t$ subject to the dilaton shift $\mathbf{q}(z) = \mathbf{t}(z) - z \in \mathcal{H}_+$ as an element in the Fock space depending on the parameter $t \in H$.

Recall that $F^1(t)$ denotes the genus 1 Gromov – Witten potential, i. e. $\mathcal{F}_X^1(\mathbf{t})$ at $t_0 = t, t_1 = t_2 = ... = 0$.

Let us introduce the operator S_t on the Laurent $1/z$-series completion of the space H defined by

$$(a, S_t b) := \langle a, \frac{b}{z - \psi} \rangle := (a, b) + \sum_{k=0}^{\infty} \langle a, b\psi^k \rangle \ z^{-1-k}.$$

The correlator notation here refers to the following 2-*point gravitational descendent* in genus 0:

$$\langle a\psi^k, b\psi^l \rangle := \sum_{m,d} \frac{Q^d}{m!} \int_{[X_{0,m+2,d}]} (\text{ev}_1^* a)\psi_1^k \ (\wedge_{i=2}^{m+1} \text{ev}_i^* t \ ) \ (\text{ev}_{m+2}^* b) \ \psi_{m+2}^l.$$

It is one of the basic facts of quantum cohomology theory (see the next section) that $S_t^*(-1/z)S_t(1/z) = \mathbf{1}$. (The asterisk here means transposition with respect to the inner product $(\cdot, \cdot)$ on H.) In other words, on a suitable completion of $(\mathcal{H}, \Omega)$, the operator S_t defines a symplectic transformation depending on the parameter $t \in H$. We put $\hat{S}_t := \exp(\ln S_t)^\wedge$.

5.1. Theorem.
$$\mathcal{D}_X = e^{F^1(t)} \hat{S}_t^{-1} \ \mathcal{A}_t.$$

5.2. *Remark.* The theorem is a reformulation of results by Kontsevich – Manin [28] computing gravitational ancestors in terms of descendents and vice versa.

Moreover, using these results, Getzler [15] proves the $3g - 2$-*jet conjecture* of Eguchi – Xiong and Dubrovin about genus g descendent potential $\mathcal{F}_X^g$ (we are not going to formulate it here — see [13] for details) by showing that it is equivalent to the following property of the genus g ancestor potentials:

$$\frac{\partial^m}{\partial t_{k_1+1}^{\alpha_1}...\partial t_{k_m+1}^{\alpha_m}} (\bar{\mathcal{F}}_t^g)|_{t_0=0} = 0 \ \ \text{if} \ \ k_1 + ... + k_m > 3g - 3.$$

The latter property is obvious: $\bar{\psi}_1^{k_1+1}...\bar{\psi}_m^{k_m+1} = 0$ since $\dim \overline{\mathcal{M}}_{g,m} = 3g - 3 + m$.

In particular, $\bar{\mathcal{F}}_t^0|_{t_0=0} = 0$. One can use this property in order to extract the genus 0 descendent potential $\mathcal{F}^0$ as follows.

5.3. Proposition. *Quantized symplectic operators of the form $S(1/z) = \mathbf{1} + S_1/z + S_2/z^2 + ...$ act on elements $\mathcal{G}$ of the Fock space as*

$$(\hat{S}^{-1}\mathcal{G})(\mathbf{q}) = e^{W(\mathbf{q},\mathbf{q})/2\hbar}\mathcal{G}([S\mathbf{q}]_+),$$

where $[S\mathbf{q}]_+$ is the power series truncation of $S(z^{-1})\mathbf{q}(z)$, and the quadratic form $W = \sum(W_{kl}q_k, q_l)$ is defined by

$$\sum_{k,l\geq 0} \frac{W_{kl}}{w^k z^l} := \frac{S^*(w^{-1})S(z^{-1}) - \mathbf{1}}{w^{-1} + z^{-1}}.$$

In particular, $\mathcal{F}_X^0(\mathbf{q}) = W_t(\mathbf{q}, \mathbf{q})/2$ if the parameter t in S_t is set to make the ancestor variable $t_0 = 0$.

5.4. Corollary. *The genus 0 descendent potential equals*

$$\mathcal{F}_X^0(\mathbf{t}) = \frac{1}{2}\langle \mathbf{t}(\psi) - \psi, \mathbf{t}(\psi) - \psi \rangle \ |_{t=t(\mathbf{t})}$$

where $t(\mathbf{t})$ is the critical point of the function $\langle \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle$ of $t \in H$ depending on the parameters $\mathbf{t} = (t_0, t_1, ...)$.

5.5. *Remarks.* The corollary is the famous reconstruction results for genus 0 gravitational descendents due to Dubrovin [6] and Dijkgraaf – Witten [5].

The "mysterious" requirement that $t = t(\mathbf{t})$ is the critical point of $\langle \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle$ arises here simply to set the argument $[S_t\mathbf{q}]_+$ in the ancestor potential equal 0 at $z = 0$.

We would like to mention the following heuristic observation (one can make on the basis of 5.3, the dual proposition 7.3 and the theorems 4.1, 5.1 and underlying geometry) about the nature of out formulas. It appears that their four basic ingredients — symplectic transformations, quantization, dilaton shift and the central charge — are governed by the four missing Deligne – Mumford spaces with $(g, m) = (0, 1), (0, 2), (0, 0)$ and $(1, 0)$.

6. Frobenius structures.

The operator series $S_t(1/z)$ introduced in the previous section is known to have the following properties [19, 21, 30].

The operator-valued 1-form $A(t) := z \ (d_t S_t(1/z)) \ S_t^{-1}(1/z)$ does not depend on z and thus defines a linear pencil of connections $\nabla_z := d - z^{-1}A(t)\wedge$ on the tangent bundle TH flat for all values of the parameter z^{-1}. The flatness condition

thus reads $A \wedge A = 0$ and $dA = 0$. In coordinates $t = \sum t^\alpha \phi_\alpha$ the first condition means commutativity $A_\alpha A_\beta = A_\beta A_\alpha$ of the components of $A = \sum A_\alpha(t) dt^\alpha$. The natural correspondence $\partial_\alpha \mapsto A_\alpha(t) = i_{\partial_\alpha} A$ defines commutative associative multiplications $\bullet_t$ on the tangent spaces $T_t H$ called the *quantum cup-product*. Its structure constants $(\phi_\alpha \bullet_t \phi_\beta, \phi_\gamma)$ actually coincide with the third directional derivatives $\partial_\alpha \partial_\beta \partial_\gamma F_X^0(t)$ of the genus 0 Gromov – Witten potential F_X^0. In particular, this explains why $dA = 0$ and also shows that $A_\alpha^* = A_\alpha$ so that the quantum cup-product is *Frobenius*: $(a \bullet b, c) = (a, b \bullet c)$. The quantum cup-product on TH is invariant under the translations in the direction of the unit element $\mathbf{1} \in H = H^*(X; \mathbf{Q}[[Q]])$ and $\mathbf{1} \bullet_t = \mathrm{id}$.

The picture just described has been axiomatized [6] under the name *Frobenius structure*. We refer to [6, 30, 19, 21, 22] for discussions of the following definition and derivations of the properties reviewed below.

6.1. *Definition.* A Frobenius structure on a manifold H consists of:
(i) a flat pseudo-Riemannian metric $(\cdot, \cdot)$,
(ii) a function F whose 3-rd covariant derivatives F_{abc} are structure constants $(a \bullet_t b, c)$ of a Frobenius algebra structure, i.e. associative commutative multiplication $\bullet_t$ satisfying $(a \bullet_t b, c) = (a, b \bullet_t c)$, on the tangent spaces $T_t H$ which depends smoothly on t;
(iii) the vector field of unities $\mathbf{1}$ of the $\bullet_t$-product which has to be covariantly constant and preserve the multiplication and the metric.

In the flat affine coordinates $t = \sum t^\alpha \phi_\alpha$ of the metric $(\cdot, \cdot)$, consider the deformation $\nabla_z := d - z^{-1} \sum (\phi_\alpha \bullet_t) \, d\mathbf{t}_\alpha \wedge$ of the Levi-Civita connection. It follows from the definition that $\nabla_z^2 = 0$ for all z. Furthermore, one can construct a fundamental solution to the system $\nabla_z S = 0$ in the form of a power series $S(1/z) = \mathbf{1} + S_1/z + S_2/z^2 + ...$ satisfying $S^*(-1/z)S(1/z) = \mathbf{1}$. Such S is unique up to right multiplication by a constant operator series $C(1/z) = \mathbf{1} + C_1/z + C_2/z^2 + ...$ satisfying $C^*(-1/z)C(1/z) = \mathbf{1}$. We will call a Frobenius manifold equipped with a choice of the solution S *calibrated*.

6.3. *The Euler field* on a Frobenius manifold is a vector field E which in a flat affine coordinate system on H has the form of a linear inhomogeneous vector field and such that $\bullet, \mathbf{1}$ and $(\cdot, \cdot)$ are eigenvectors of the Lie derivative along E with the eigenvalues $0, -1$ and $2 - D$ respectively. The Frobenius manifold equipped with an Euler vector field is called *conformal* of conformal dimension D. We will require that calibrations S of conformal Frobenius structures are homogeneous in the sense that the bilinear form $(a, S(1/z)b)$ on TH is an eigenvector of $z\partial_z + E$ with the eigenvalue $2 - D$. This reduces the choice of S to finitely many constants, and we refer to [7] for precise description of the ambiguity.

Following [7], we will assume the linear part of the Euler field semisimple.

6.4. *Examples.* (a) According to K. Saito [34] orbit spaces of finite irreducible Coxeter groups carry canonical conformal Frobenius structure.

(b) Miniversal deformations of isolated critical points of holomorphic functions can be naturally equipped with conformal Frobenius structures [35] (see also [30, 18, 22]).

(c) Genus 0 Gromov – Witten invariants of a compact almost Kähler manifold X define on $H = H^*(X; \mathbf{Q}[[Q]])$ a formal structure of a calibrated conformal Frobenius manifold of conformal dimension $D = \dim_{\mathbf{C}} X$. In the coordinate system $t =$

$\sum t^\alpha \phi_\alpha$ corresponding to a graded basis $\{\phi_\alpha\}$ in $H^*(X, \mathbf{Q})$, the Euler field takes on the form

$$E = \sum (1 - \deg \phi_a / 2) t_\alpha \partial_\alpha + \rho$$

where the constant part $\rho \in H^*(X)$ is the 1-st Chern class of the tangent bundle TX.

(d) In the section 9 we will deal with equivariant generalization [19, 37] of Gromov – Witten theory in the case when X is equipped with a hamiltonian action of a compact Lie group G. In this case equivariant genus 0 Gromov – Witten invariants define on equivariant cohomology space $H = H_G^*(X; \mathbf{Q}[[Q]])$ the structure of a calibrated Frobenius manifold. It is not conformal though since the Euler field defined by cohomology grading over $\mathbf{Q}[[Q]]$ is not a derivation over the coefficient ring $H_G^*(pt) = H^*(BG)$ of equivariant cohomology theory.

6.5. *Remark.* We will apply our formalism of quantized quadratic hamiltonians to the general problem of equipping Frobenius manifolds with all attributes of higher genus Gromov – Witten theory. The function F from the definition 6.1 is taken of course on the role of the genus 0 potential F^0. The calibration S is then used to define the 1-point descendent $\langle a, b/(z-\psi) \rangle := (a, S(1/z)b)$. Then Corollary 5.4 yields Dubrovin's construction [6] of the genus 0 descendent potential $\mathcal{F}^0$.

Below we give a construction of the total descendent potentials $\mathcal{D}$ in the case when the Frobenius manifold is *semisimple* i. e. when Frobenius algebras $(T_t H, \bullet_t)$ are semisimple at generic $t \in H$. All Frobenius manifolds of Examples $6.4a, b$ are semisimple as well as those in d when G is a torus acting on X with isolated fixed points. Among examples $6.4c$ flag manifolds, toric Fano manifolds and probably many other Fano manifolds yield semisimple Frobenius structures.

6.6. *Canonical coordinates.* Let $t = \sum t^\alpha \phi_\alpha$ be a flat coordinate system on the Frobenius manifold H. Let $u \in H$ be a semisimple point so that the matrices $A_\alpha = \phi_\alpha \bullet_u$ are simultaneously diagonalizable. Let Ψ be the transition matrix from the basis $\{\phi_\alpha\}$ in $T_u H$ to the basis of common eigenvectors of A_α normalized to the unit lengths. Then $\Psi^{-1} A^1 \Psi$ is a diagonal matrix of *closed* 1-forms and has locally the form $dU = \mathrm{diag}(du^1, ..., du^N)$. The diagonal entries of the potential matrix $U = \mathrm{diag}(u^1, ..., u^N)$ form a local coordinate system on H called *canonical* [6]. Canonical coordinates are defined uniquely up to signs, permutations and additive constants which are set to zero in the conformal case by the requirements $Eu^i = u_i$.

6.7. **Proposition** [22, 21, 6]. *(a) The equation $\nabla_z S = 0$ in a neighborhood of a semisimple point u has a fundamental solution in the form: $\Psi_u R_u(z) e^{U/z}$, where $R_u(z) = \mathbf{1} + R_1 z + R_2 z^2 + ...$ is a formal matrix power series satisfying $R_u^*(-z) R_u(z) = \mathbf{1}$.*

(b) The series $R_u(z)$ satisfying (a) is unique up to right multiplication by diagonal matrices $\exp(a_1 z + a_2 z^3 + a_3 z^5 + ...)$ where $a_k = \mathrm{diag}(a_k^1, ..., a_k^N)$ are constant.

(c) In the case of conformal Frobenius structures the series $R_u(z)$ satisfying (a) is uniquely determined by the homogeneity condition $(z\partial_z + \sum u^i \partial_{u^i}) R_u(z) = 0$.

Let $\mathcal{H}$ be the space of Laurent polynomials in z with coefficients in the tangent space $T_u H$ to the Frobenius manifold at a semisimple point u. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be the polarization of $\mathcal{H}$ described in section 2. For $\mathbf{q}(z) \in \mathcal{H}_+$ we write $\Psi_u^{-1} \mathbf{q}(z) = (\mathbf{q}^1(z), ..., \mathbf{q}^N(z))$ and introduce the direct product $\mathcal{T} = \tau(\hbar; \mathbf{q}^1) ... \tau(\hbar; \mathbf{q}^N)$ of N copies of the Witten – Kontsevich tau-function as an element of the Fock space of functions on $\mathcal{H}_+$. Next, the series $R_u(z)$ defines a symplectic transformation

on $\mathcal{H}$, and we put $\hat{R}_u = \exp(\ln R_u)\hat{\ }$. Slightly abusing notation we denote $\hat{\Psi}_u$ the operator $\mathcal{G}(\Psi^{-1}\mathbf{q}) \mapsto \mathcal{G}(\mathbf{q})$ identifying the Fock space with its coordinate version. Finally, $\sum_{i=1}^{N} R_1^{ii} du^i$ where R_1^{ii} are the diagonal entries of the matrix R_1 in the series $R_u(z) = \mathbf{1} + R_1/z + ...$ is known to be closed as a 1-form on the Frobenius manifold. We introduce the function $C(u) = \frac{1}{2} \int^u \sum R_1^{ii} du^i$ of u defined up to an additive constant.

6.8. *Definition.* We define the *total descendent potential* of a semisimple Frobenius manifold by the formula

$$\mathcal{D}(\hbar; \mathbf{t}) \; := \; e^{C(u)} \; \hat{S}_u^{-1} \; \hat{\Psi} \; \hat{R}_u \; e^{(U/z)\hat{\ }} \; \mathcal{T}.$$

We introduce the *total ancestor potential* of a semisimple Frobenius manifold:

$$e^{F^1(u)} \; \mathcal{A}_u(\hbar; \mathbf{t}) \; := \; e^{C(u)} \; \hat{\Psi} \; \hat{R}_u \; e^{(U/z)\hat{\ }} \; \mathcal{T}.$$

Since $\mathcal{D}$ and $\mathcal{A}_u$ are related as in Theorem 5.1, these definitions automatically agree with the reconstruction formula 5.4 for $\mathcal{F}^0$. We keep the convention about the dilaton shift 2.2 in these definitions.

6.9. *Remarks.* (a) Both potentials depend on a choice of the asymptotical series $R_u(z)$ (which is unambiguous in the conformal case) and are defined up to a constant factor *independent on u*. In section 9 we will specify the choice of $R_u(z)$ corresponding to equivariant Gromov – Witten theory.

(b) The factor $\exp(U/z)\hat{\ }$ is redundant since the string operator $(1/z)\hat{\ } = \hat{L}_{-1}$ annihilates the (product of) Witten – Kontsevich tau-functions as it follows from 3.1 and is included only for future convenience.

(c) Quantizations of symplectic transformations $S_u(1/z)$ and $R_u(z)$ require different power series completion of the space $\mathcal{H}_+$ so that their composition, strictly speaking, is not defined. Nevertheless the formula for $\mathcal{D}$ makes sense (in particular — due to some properties of the function $\mathcal{T}$) at least in the formal neighborhood of the critical point locus $u = t(\mathbf{t})$ described in 5.4. As a result of this subtlety, the potentials of abstract Frobenius structures thus defined do not always extend to non-semisimple values of $t_0 = t(\mathbf{0})$. We are not going to stress this subtlety in the rest of the text.

7. Properties of the total descendent potential.

7.1. Theorem. *The total descendent potential $\mathcal{D}$ of a semisimple Frobenius manifold defined in* 6.8 *does not depend on the choice of a semisimple point u.*

This follows from the property of $S_u(1/z)$ and $\Psi_u R_u(z) \exp(U/z)$ to satisfy the same differential equation $\nabla_z S = 0$ with coefficients rational in z. As a result of this, derivatives of $\mathcal{D}$ in the directions of the parameter u vanish as if S_u^{-1} and $\Psi_u R_u \exp(U/z)$ were inverse to each other. The factor $e^{C(u)}$ however is needed in order to offset the effect of the cocycle $\mathcal{C}$ under quantization of this computation.

7.2. Corollary. *(a) The genus* 1 *Gromov – Witten potential $F^1(t)$ of a semisimple Frobenius manifold is given by the formula*

$$F^1(t) = C(u) - \frac{1}{48} \ln \det(\partial_{u^i}, \partial_{u^j}) = \frac{1}{2} \int^u \sum R_1^{ii} du^i + \frac{1}{48} \sum \ln \Delta_i,$$

where $\Delta_i^{-1}(u) = (\partial_{u^i}, \partial_{u^i})$ are inner squares of the canonical idempotents ∂_{u^i} of the semisimple Frobenius multiplication $\bullet_u$.

(b) The genus 1 descendent potential equals

$$\mathcal{F}_X^1(\mathbf{t}) = F^1(t(\mathbf{t})) \; + \; \frac{1}{24} \ln \det[\partial_{t_0^\alpha} \partial_{t_0^\beta} \partial_{t_0^0} \mathcal{F}_X^0(\mathbf{t})],$$

where the partial derivatives are taken with respect to coordinates of $t^0 = \sum t_0^\alpha \phi_\alpha$.

The part (a) coincides with the conjectural formula for F_X^1 suggested in [21] and proved in [8] in the conformal case. A similarly looking formula can be derived for the genus 1 descendent potential. It is shown in [22], Example 8, how to reconcile it with the well-known formula in part (b) which is due to Dijgraaf – Witten [5] and does not require semisimplicity hypotheses.

7.3. Proposition. *Quantized symplectic operators of the form $R(z) = 1 + R_1 z + R_2 z^2 + ...$ act on elements $\mathcal{G}$ of the Fock space as*

$$(\hat{R}\mathcal{G})(\mathbf{q}) = [e^{\hbar V(\partial_\mathbf{q}, \partial_\mathbf{q})/2} \; \mathcal{G}](R^{-1}\mathbf{q}),$$

where $R^{-1}\mathbf{q}$ is defined as the product of z-series $R^{-1}(z)\mathbf{q}(z)$, and the quadratic form $V = \sum(p_k, V_{kl} p_l)$ is defined by

$$\sum_{k,l \geq 0} (-1)^{k+l} V_{kl} w^k z^l = \frac{R^*(w)R(z) - \mathbf{1}}{w + z}.$$

7.4. Corollary. *The total descendent potential $\mathcal{D}$ of a semisimple Frobenius manifold satisfies the $3g - 2$-jet conjecture of Eguchi – Xiong (see Remark 5.2).*

This time the property of genus g ancestor potentials

$$\frac{\partial^m}{\partial_{t_{k_1+1}^{\alpha_1}} ... \partial_{t_{k_m+1}^{\alpha_m}}} \; (\bar{\mathcal{F}}_u^g)|_{t_0=0} \; = \; 0 \; \text{ if } \; k_1 + ... + k_m > 3g - 3$$

follows from the dimensional properties of the product $\mathcal{T}$ of Witten-Kontsevich tau-functions and from the structure ("upper-triangular" in some sense) of the operators $\hat{R}$ described in the proposition.

Propositions 5.3 and 7.3 together imply:

7.5. Corollary. *The definition 6.8 of the descendent potential of a semisimple Frobenius manifold agrees with the construction of genus > 1 (descendent) potentials $F^g(t)$ (respectively $\mathcal{F}^g(\mathbf{t})$) suggested in [22].*

7.6. Conjecture. *$\mathcal{D}_X = \mathcal{D}$: The total descendent potential $\mathcal{D}$ corresponding to the calibrated conformal Frobenius structure on the cohomology space $H = H^*(X; \mathbf{Q}[[Q]])$ defined by genus 0 Gromov – Witten invariants of a compact almost Kähler manifold X with generically semisimple quantum cup-product coincides with the total descendent potential $\mathcal{D}_X$ defined in the Gromov – Witten theory of X.*

This conjecture is equivalent to Conjecture 2 in [22]. In sections $9, 10$ we outline a proof of this conjecture for complex projective spaces and other toric Fano manifolds.

The Virasoro operators $\hat{L}_m + N\delta_{m,0}/16$ from the section 3 annihilate the product $\mathcal{T}$ of N Witten – Kontsevich tau-functions. Let us put formally

$$\mathcal{L}_m := \hat{S}_u^{-1} \hat{\Psi}_u \hat{R}_u \; \hat{L}_m \; \hat{R}_u^{-1} \hat{\Psi}_u^{-1} \hat{S}_u.$$

7.7. Proposition. *The operators $\mathcal{L}_m + N\delta_{m,0}/16$ satisfy the commutation relations $[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n}$ of the algebra of vector fields on the line and*

annihilate the total descendent potential of the semisimple Frobenius manifold:
$\mathcal{L}_m\mathcal{D} = 0, \ m = -1, 0, 1, 2, ...$

The formal Virasoro constraints described in the proposition will be computed in the next section in the case of conformal Frobenius structures.

8. Virasoro constraints.

Consider a Frobenius structure of conformal dimension D with the Euler vector field which is the sum of a linear diagonalizable and a constant vector field. Then in suitable flat coordinates of the metric it assumes the form

$$E = \sum(1 - d_\alpha)t^\alpha\partial_{t^\alpha} + \sum_{\alpha:\ d_\alpha=1}\rho_\alpha\partial_{t^\alpha},$$

where the degree spectrum $\{d_\alpha\}$ is confined on the interval $[0, D]$ and is symmetric about $D/2$. The homogeneity conditions for the calibration $S(1/z)$ and the asymptotical solution $T = \Psi R(z)e^{U/z}$ read respectively:

$$(z\partial_z + E)S = \mu S + S(\mu + \rho/z)^*, \quad (z\partial_z + E)T = \mu T,$$

where $\mu = \mathrm{diag}(d_1 - D/2, ..., d_N - D/2)$ is anti-symmetric ($\mu = -\mu^*$), and $\rho = \rho^*$ is μ-*nilpotent* in the sense of [7]. In Gromov – Witten theory, μ is the *Hodge grading operator* and ρ is the operator of multiplication by $c_1(TX) = \sum\rho_\alpha\partial_{t^\alpha}$ in the *classical* cohomology algebra.

8.1. Theorem. *The Virasoro operators* 7.7 *are* $\mathcal{L}_m = \hat{L}_m^{\mu,\rho} + \frac{\delta_{m,0}}{4}\,\mathrm{tr}(\mu\mu^*)$, *where* $L_m^{\mu,\rho}$ *are the infinitesimal symplectic transformations*

$$L_m^{\mu,\rho} = z^\mu z^{-\rho}L_m z^\rho z^{-\mu} = z^{-1/2}(z\frac{d}{dz}z - \mu z + \rho)^{m+1}z^{-1/2}.$$

The Virasoro operators 8.1 actually coincide with those introduced in [9].

Regardless of the validity of the conjecture 7.6, our definition of the total descendent potential $\mathcal{D}$ reproduces correctly the formulas 5.4 and $5.5 + 7.2$ for genus 0 and 1 descendent potentials in Gromov – Witten theory. As a consequence, we obtain a new proof of the main result in [9]:

8.2. Corollary. *The genus* 1 *Virasoro constraints hold true for Gromov – Witten invariants of almost Kähler manifolds with generically semisimple quantum cup-product.*

8.3. *Remarks.* (a) Put $w = 1/z$. Given a connection operator $D = d/dw + A(w)/w$ where $A = A_0w + A_1w^2 + ...$, $A^*(-w) = -A(w)$, one can associate to it a representation of the Lie algebra of vector fields on the line to the algebra of quadratic hamiltonians on $\mathcal{H}$ by taking $L_m^D := w^{3/2}D^{m+1}w^{-1/2}$. Quantization then yields a representation in the Fock space. If two connections are conjugated by a gauge transformation $D \mapsto S^{-1}DS$ where $S(w) = S_0 + S_1w + ...$, $S^*(-w)S(w) = \mathbf{1}$, then the corresponding representations are equivalent. According to [7], the classes of gauge equivalence (in the case when the residue operator A_0, $A_0 = -A_0^*$, is semisimple) are represented by the connections of the form $d/dw + (\mu - w\rho)/w$ where ρ is polynomial in w and is μ-*nilpotent* (see [7]). The representation in the theorem corresponds to D in the normal form with constant ρ.

(b) In the case of conformal Frobenius structures the consistent PDE system $d\Phi = z^{-1}A(t)\Phi$ on H can be completed, following [6], to a consistent PDE system

on $\mathbf{C} \times H$ by adding the homogeneity equation $(d/dw + \mu/w - E\bullet_t)\Phi = 0$. The latter equation can be considered as a family D_t of connections on $\mathbf{C}$ depending on the parameter $t \in H$ and is *isomonodromic*. It is the role of the calibration $S_t(w)$ to conjugate the connection operators D_t to the normal form $d/dw + \mu/w - \rho$ with the fundamental solution $w^{-\mu}w^\rho$. In other words, $\Phi = S_t(w)w^{-\mu}w^\rho$ is a fundamental solution to the PDE system on $\mathbf{C} \times H$. The key point in the proof of the theorem 8.1 is that the asymptotical solution T satisfies the same equations as Φ and thus $S^{-1}T$ satisfies the same equations as $w^{-\mu}w^\rho$.

9. Equivariant Gromov – Witten theory.

Let the compact almost Kähler manifold X be equipped with a hamiltonian Killing action of a compact Lie group G. Then the moduli spaces $X_{g,m,d}$ inherit the action, and the evaluation, forgetting and contraction maps are G-equivariant. The construction of the virtual fundamental class $[X_{g,m,d}]$ admits an equivariant generalization [33]. This allows one to introduce G-equivariant counterparts of Gromov – Witten invariants, gravitational descendents and ancestors and of the generating functions $F_X^g, \mathcal{F}_X^g, \mathcal{D}_X, \mathcal{A}_X$, etc. The invariants take values in $H_G^*(point; \mathbf{Q}) = H^*(BG; \mathbf{Q})$, the coefficient ring of equivariant cohomology theory. They reduce to their non-equivariant versions under the restriction homomorphism $H^*(BG; \mathbf{Q}) \to H^*(point; \mathbf{Q})$.

The genus 0 equivariant Gromov – Witten potential F_X^0 defines on the equivariant cohomology space $H = H_G^*(X; \mathbf{Q}[[Q]])$ the structure of a formal Frobenius manifold *over the ground ring* $H_G^*(point; \mathbf{Q})$ (see [19]).

Let us consider the case when G is a torus $(S^1)^r$. The cohomology algebra of the classifying space $BG = (\mathbf{C}P^\infty)^r$ is isomorphic to $\mathbf{Q}[\lambda_1, ..., \lambda_r]$. Here $\lambda_i \in Lie^*G$ are identified with a basis of infinitesimal characters of 1-dimensional representations of $(S^1)^r$. The representations induce Hopf line bundles over $(\mathbf{C}P^\infty)^r$ whose Chern classes generate $H^*(BG)$.

When the fixed points of the torus G action on X are isolated, the Frobenius structure on H is semisimple. Indeed, even the classical equivariant cohomology algebra of X with coefficients in the field of fractions $\mathbf{Q}(\lambda)$ of the ground ring $H_G^*(point; \mathbf{Q})$ is semisimple. Therefore the most of results of the previous sections apply in equivariant Gromov – Witten theory. However, the Frobenius structure in question is not conformal: the Euler vector field expressing dimensional properties of Gromov – Witten invariants is a derivation over $\mathbf{Q}$ but, generally speaking, not over the ground ring $H_G^*(point; \mathbf{Q})$ since elements of this ring may have non-zero degrees. As a consequence, the part (c) of Proposition 6.7 does not apply, i. e. the construction 6.7(a) of the series $R(z)$ is ambiguous, and the ambiguity is as described in 6.7(b).

Let w_i, $i = 1, ..., N$ be the fixed points of the torus G action on X, and $N_l^{(i)}$ denote the Newton polynomial $\sum_{j=1}^n \chi_j^{-l}(w_i)$ of the inverse infinitesimal characters $\chi_j^{-1}(w_i)$ of the torus G action on the cotangent space $T_{w_i}^*X$ at the fixed point. We consider $N_l^{(i)}$ as an element of the ground field $\mathbf{Q}(\lambda_1, ..., \lambda_r)$.

9.1. Theorem. *In the case of a torus G action with isolated fixed points, the total ancestor potential in the equivariant Gromov – Witten theory on X coincides with the total ancestor potential 6.8 of the semisimple Frobenius structure on $H = H_G^*(X; \mathbf{Q}[[Q]])$ (i. e. $e^{F_X^1}\mathcal{A}_X = e^C \hat{\Psi}\hat{R}\mathcal{T}$) provided that the series $R(z)$ is normalized*

by the condition that in the classical cohomology limit $Q \to 0$ it takes on the form
$R(z)|_{Q=0} = \exp \mathrm{diag}(b_1, ..., b_N)$ *where b_i are the* Bernoulli series

$$b_i(z) = \sum_{k=1}^{\infty} N_{2k-1}^{(i)} \frac{B_{2k}}{2k} \frac{z^{2k-1}}{2k-1}.$$

9.2. *Remarks.* (a) The theorem is equivalent (due to 5.1) to the Theorem 2 in [22] describing descendent potentials $\mathcal{F}_X^g$ with $g > 1$ (plus a similar result of [21] about F_X^1). Both results are proved by fixed point localization in moduli spaces $X_{g,m,d}$ utilizing Kontsevich's technique of summations over graphs [27]. Justification of fixed point localization formulas for virtual fundamental classes depends on the results of [23] in the algebraic category and [37] in the general almost Kähler setting. The Bernoulli series arise to offset the effect of Hodge integrals in localization formulas by applying the theorem 4.1 for $X = point$.

(b) It is essential in the formulation 9.1 that indices of canonical coordinates coincide with the labels $i = 1, ..., N$ of fixed points. This is due to the phenomenon [19, 21] of *materialization* of canonical coordinates in fixed point localization theory, playing an essential role in the proofs [21, 22] otherwise too. The simplest instance of it is related to the equality

$$\sum u^i = \text{ the total "number" of elliptic curves in } X \text{ with given modulus.}$$

In the computation of the "number" via the fixed point technique, we can single out contributions v^i of those fixed curves where *the* elliptic irreducible component (with the required generic modulus) is mapped to the fixed point w_i. It turns out [19] that the equality $\sum v^i = \sum u^i$ is *termwise* (which in particular establishes the aforementioned correspondence).

(c) The ancestor potential $\mathcal{A}_X$ specializes to its non-equivariant counterpart in the non-equivariant limit $\lambda = 0$. It is not clear however how to derive from the theorem its non-equivariant version. It would suffice to show that the series $R(z)$ tends to its non-equivariant counterpart (since all other ingredients of the formula obviously do). Yet, if for some normalization of $R(z)$ the limit exists, then it does not exist for any other normalization.

9.3. Conjecture. *When X has generically semisimple non-equivariant quantum cohomology, the series $R(z)$ (normalized by the Bernoulli series as in 9.1) has a non-equivariant limit at $\lambda = 0$ equal to the series R from 7.6.*

In the next section we prove this conjecture in the case of complex projective spaces.

10. Mirrors of complex projective spaces.

According to the theorem 7.1 the total descendent potential $\mathcal{D}$ of a semisimple Frobenius manifold is determined by the values of S, Ψ and R at any semisimple point. We will exploit the property of projective spaces and their products to have semisimple *small* quantum cohomology algebra.

Let $G = T^{n+1}$ be the torus of diagonal unitary transformations of the standard Hermitian space $\mathbf{C}^{n+1}$. We identify $H^*(BG, \mathbf{Q})$ with $\mathbf{C}[\lambda_0, ..., \lambda_n]$ and the equivariant cohomology algebra $H_G^*(\mathbf{C}P^n, \mathbf{Q})$ of the projectivized space — with $\mathbf{C}[P, \lambda_0, ..., \lambda_n]/((P - \lambda_0)...(P - \lambda_n))$. Here $-P$ denote the equivariant 1-st Chern class of the Hopf line bundle over $\mathbf{C}P^n$. The multiplication table in the basis

$1, P, ..., P^{n-1}$ in the *small* equivariant quantum cohomology algebra of $\mathbf{C}P^n$ has the following well-known description:

$$(P - \lambda_0) \bullet ... \bullet (P - \lambda_n) = Q, \quad \text{while} \quad P \bullet P^k = P^{k+1} \text{ for } k < n.$$

Here the generator Q in the Novikov ring is identified with $\exp t$ where t is the coordinate on the parameter space $H^2(\mathbf{C}P^n, \mathbf{C})$ of the *small* quantum cohomology algebra. (The identification is possible due to the *divisor equation*, see for instance [19].) Respectively, the connection ∇_z from 6.7 restricted to this parameter space yields the following system of linear differential equations:

$$z\frac{d}{dt}I_k = I_{k+1} \text{ for } k < n, \text{ and } \quad (z\frac{d}{dt} - \lambda_0)...(z\frac{d}{dt} - \lambda_n)\, I_0 \;=\; e^t \, I_0.$$

Introduce the complex oscillating integral

$$I = \int_{\Gamma \subset \{x_0...x_n = Q\}} e^{(x_0 + ... + x_n)/z}\; x_0^{\lambda_0/z} ...\; x_n^{\lambda_n/z}\; \frac{dx_0 \wedge ... \wedge dx_n}{d(x_0...dx_n)}.$$

The cycles Γ in the oscillating integrals of the form $\int_\Gamma e^{F(x)/z}\phi(x)dx$ may be non-compact and are constructed by means of Morse theory for the functions $\mathrm{Re}(F/z)$.

10.1. Theorem. $(zQ\frac{d}{dQ} - \lambda_0)...(zQ\frac{d}{dQ} - \lambda_n)\, I \;=\; Q\, I.$

Proof. It is convenient to rewrite the integral in logarithmic coordinates: $Q = e^t$, $x_i = e^{T_i}$:

$$I = \int_{\Gamma \subset \{t = \sum T_i\}} e^{\sum(e^{T_i} + \lambda_i T_i)/z}\; \frac{dT_0 \wedge ... \wedge dT_n}{d(T_0 + ... + T_n)}.$$

The projection $(T_0, ..., T_n) \mapsto t = T_0 + ... + T_n$ maps each ∂_{T_i} to ∂_t. Using this we conclude that application of $zd/dt - \lambda_i$ to the integral can be replaced by multiplication of the integrand by e^{T_i}. Doing this consecutively for $i = 0, 1, ..., n$ we find the integrand multiplied by $e^{T_0}...e^{T_n} = e^t$. Thus $(zd/dt - \lambda_0)...(zd/dt - \lambda_n)I = e^t I$.

10.2. Remark. The result is an equivariant version of the mirror theorem for complex projective spaces we reported in Summer 93 at seminars in Lyon, Strasbourg and Oberwolfach. Many mathematical results inspired by or predicted on the basis of mirror theory have been proved since then. Yet the argument presently discussed seems to be one of the first formal mathematical applications of a mirror theorem.

An asymptotical solution $T = \Psi R(z) \exp(U/z)$ to the system $\nabla_z T = 0$ can be obtained from stationary phase asymptotics near the critical points of the phase function of the integral I and its derivatives.

10.3. Example. We illustrate the procedure of expanding a complex oscillating integral near a non-degenerate critical point of the phase function with the critical value u:

$$\int e^{(u - x^2 + \alpha x^3 + ...)/z}(\beta + \gamma x + ...)dx = \sqrt{z}e^{u/z}\int e^{-y^2}e^{\alpha y\sqrt{z} + \cdots}(\beta + \gamma y\sqrt{z} + ...)dy.$$

Discarding the dimensional factor $\sqrt{z}$ in front of the integral and computing momenta of the Gaussian distribution yields an asymptotical expansion of the form $e^{u/z}\,(a + bz + cz^2 + ...)$.

In the oscillating integral I, the critical points of the phase function are constrained extrema of the function

$$e^{T_0} + ... + e^{T_n} + \lambda_0 T_0 + ... + \lambda_n T_n - P(T_0 + ... + T_n - t)$$

with the Lagrange multiplier P. They satisfy $e^{T_\alpha} = P - \lambda_i$ where $(P - \lambda_0)...(P - \lambda_n) = Q$. Near $Q = 0$ and generic λ, the $n+1$ roots $P^{(i)}(Q)$, $i = 0, ..., n$, to the latter equation are distinguished by their values $P^{(i)}(0) = \lambda_i$. On the other hand the basis $\{P^k, k = 0, ..., n\}$ in $H^*(\mathbf{C}P^n)$ corresponds to the basis of integrals $I_k = z(d/dt)^k I$ (since $P^{\bullet k} = P^k$ for $k \leq n$). In the asymptotical solution $\Psi R(z) \exp(U/z)$ the diagonal matrix U consists of the critical values u^i, and the matrix entries of $\Psi R(z)$ are obtained from asymptotical expansions of I_k near the critical point $P^{(i)}$.

The asymptotical solution thus constructed automatically admits the non - equivariant limit $\lambda = 0$ when $Q \neq 0$ since the phase functions, amplitudes and the critical points in the oscillating integrals depend continuously on λ. For generic λ, the solution depends continuously on Q up to $Q = 0$.

At $Q = 0$ the matrix Ψ describes the transition from the basis $\{P^k\}$ to the basis $L_j(P) = \prod_\alpha (P - \lambda_\alpha)/(\lambda_j - \lambda_\alpha)^{1/2}$ of (suitably normalized) Lagrange interpolation polynomials. Indeed, the Lagrange interpolation polynomials represent the basis of delta-functions of of the fixed points $w_j, j = 0, ..., n$, in the equivariant cohomology algebra of $\mathbf{C}P^n$ and therefore coincide with the canonical idempotents of the semisimple algebra $(T_u H, \bullet_u)$ in the classical cohomology limit $Q = 0$. Thus the matrix elements of the series $R(z)$ at $Q = 0$ are extracted from asymptotical expansions near the critical points $P^{(i)}$ of the integrals $L_j(zd/dt)I$ in this limit.

For symmetry reasons it suffices to consider only the critical point corresponding to $P^{(0)}$. Computing $L_j(zd/dt)I$ as in the proof of 10.1 we find an extra factor proportional to Q/x_j in the integrand of the oscillating integral I. We rewrite the integral in the chart $(x_1, ..., x_n)$ where $x_0 = Q/x_1...x_n$. It still contains $P^{(0)}$ in the limit $Q = 0$ when $x_0 = P^{(0)} - \lambda_0$ vanishes. We find $L_j(zd/dt)I$ proportional to

$$e^{\lambda_0 \ln Q/z} \int e^{[\frac{Q}{x_1...x_n} + x_1 + ... + x_n + (\lambda_1 - \lambda_0)\ln x_1 + ... + (\lambda_n - \lambda_1)\ln x_n]/z} \frac{Q}{x_j} \frac{dx_1 \wedge ... \wedge dx_n}{x_1...x_n}.$$

We see that the integral vanishes at $Q = 0$ unless $j = 0$. This agrees with the fact of our general theory that R is diagonal at $Q = 0$. For $j = 0$ we have $Q/x_0 = x_1...x_n$, and the integral at $Q = 0$ factors into 1-dimensional ones:

$$\int e^{(\sum_{i=1}^n x_i + (\lambda_i - \lambda_0)\ln x_i)/z} \ dx_1 \wedge ... \wedge dx_n = \prod_{i=1}^n \int e^{x/z} x^{(\lambda_i - \lambda_0)/z} dx.$$

The stationary phase asymptotics of the latter integral is the same as for the product $\Gamma(1 + (\lambda_1 - \lambda_0)/z)...\Gamma(1 + (\lambda_n - \lambda_0)/z)$ of the gamma-functions as $z \to 0$ since

$$\int e^{x/z} \ x^{\lambda/z} \ dx = (-z)^{1+\lambda/z} \int e^{-v} \ v^{\lambda/z} \ dv \ \sim \ \Gamma(1 + \frac{\lambda}{z}).$$

10.4. Lemma. $\ln \Gamma(1 + s) \sim s \ln(s/e) + \frac{1}{2} \ln(2\pi s) + \sum_{k=1}^\infty \frac{B_{2k}}{2k} \frac{s^{1-2k}}{2k-1}.$

Proof. This is well-known, see for instance [24].

Subtracting the critical value (Stirling's approximation) we find that at $Q = 0$

$$\ln R^{00} = \sum_{k=1}^\infty \frac{B_{2k}}{2k} \frac{z^{2k-1}}{2k-1} \sum_{\alpha \neq 0} (\lambda_\alpha - \lambda_0)^{1-2k}.$$

This coincides with $b_0(z)$ in 9.1 since $\lambda_\alpha - \lambda_0$ are exactly the infinitesimal characters of the torus G action on $T^*_{w_0}\mathbf{C}P^n$.

Finally, our computation carries over without significant changes to arbitrary Fano toric manifolds X. It is essential here that (i) Fano toric manifolds have semisimple *small* quantum cohomology (as it follows from the explicit description of such cohomology given by V. Batyrev [1]) and (ii) *equivariant* genus 0 GW-invariants of Fano toric manifolds admit a mirror description [21] by complex oscillating integrals generalizing Theorem 10.1.

Combining the computation with 5.1 and 9.1 (or equivalently — with [22]) we prove the conjecture 7.6.

10.5. Theorem. *The total descendent potential of a toric Fano manifold X is given, up to a non-zero constant factor, by the formula $\mathcal{D}_X = e^{C(t)}\hat{S}_t\hat{\Psi}_t\hat{R}_t\mathcal{T}$ where $t \in H^2(X; \mathbf{C})$.*

10.6. Corollary. *The total descendent potential $\mathcal{D}_X$ of a toric Fano manifold satisfies $(\mathcal{L}_m + N\delta_{m,0}/16)\mathcal{D}_X = 0$ for $m = -1, 0, 1, 2, ...$, where $\mathcal{L}_m$ are the Virasoro operators 8.1.*

10.7. *Remarks.* (a) The corollary was conjectured by T. Eguchi, K. Hori, M. Jinzenji, C.-S. Xiong and S. Katz [11, 12].

(b) The so called λ_g-*conjecture* and some other non-trivial properties of Hodge integrals over Deligne – Mumford spaces were discovered by Getzler and Pandharipande [16] on the basis of Virasoro conjecture for $\mathbf{C}P^1$. The Corollary therefore backs up the original arguments of [16] which appear to be more natural than the proof of the properties found later by Faber and Pandharipande [14].

(c) The role the gamma-function 10.4 plays in a general theorem 9.1 should probably remind us about the relationship of quantum cohomology theory with its heuristic roots [17, 18] — S^1-equivariant Floer theory on loop spaces. In fact the gamma-function 10.4 will make a more systematic appearance in [4] in the context of general "quantum" versions of the Riemann – Roch formula, Serre duality and Lefschetz hyperplane section theorem.

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