A mirror theorem for toric complete intersections

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Abstract

We prove a generalized mirror conjecture for non-negative complete intersections in symplectic toric manifolds. Namely, we express solutions of the PDE system describing quantum cohomology of such a manifold in terms of suitable hypergeometric functions.

0. Introduction. Let X denote a non-singular compact Kähler toric variety with the Picard number k. The variety X can be obtained by the symplectic reduction of the standard Hermitian space $\mathbb{C}^N$ by the action of a subtorus T^k in the torus T^N of diagonal unitary matrices on a generic level of the momentum map (see Section 3). The coordinate hyperplanes in $\mathbb{C}^N$ are T^N-invariant, and their T^k-reductions on the same level of the momentum map define N compact toric hypersurfaces in X. We denote $u_1, ..., u_N$ the classes in $H^2(X)$ Poincare-dual to the fundamental cycles of these hypersurfaces. It is known that $H^2(X)$ is a free abelian group of rank k spanned by $u_1, ..., u_N$, that it multiplicatively generates the ring $H^*(X)$, and that the 1-st Chern class $c(\mathcal{T}_X)$ of the tangent bundle to X is equal to $u_1 + ... + u_N$.

Let us consider the sum $\mathcal{V}$ of $l \geq 0$ non-negative line bundles over X with the 1-st Chern classes $v_1, ..., v_l$ and denote Y the non-singular complete intersection in X of dimension $N - k - l$ defined by global holomorphic sections of these line bundles. The inclusion $Y \subset X$ induces the homomorphism

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$H^2(X) \to H^2(Y)$ The cohomology ring $H^*(Y)$ contains the subring multiplicatively generated by the classes $u_1, ..., u_N$. We denote this subring $H^*(\mathcal{V})$. It carries the Poincare pairing $\langle f, g \rangle = \int_{[Y]} fg = \int_{[X]} fg v_1...v_l$ non-degenerate over $\mathbb{Q}$. The 1-st Chern class $c(\mathcal{T}_Y)$ of the tangent bundle to Y is equal to $u_1 + ... + u_N - v_1 - ... - v_l$.

Given a compact Kähler manifold Y, the Gromov - Witten theory [14] associates to it the *quantum cohomology algebra* and the *quantum cohomology $\mathcal{D}$-module*.

Let us introduce the semigroup $\Lambda \subset H_2(Y)$ of fundamental classes of compact holomorphic curves in Y. The quantum cohomology algebra is a skew - commutative associative deformation of the cohomology algebra $H^*(Y, \mathbb{Q}[[\Lambda]])$ with coefficients in a formal completion of the semigroup algebra $\mathbb{Q}[\Lambda]$. The structural constants $\langle a \circ b, c \rangle$ of the quantum multiplication $\circ$ are formal series $\sum_{d \in \Lambda} n_d(a, b, c) q^d$ where q^d is the element of the semigroup ring corresponding to the homology class $d \in \Lambda$. The coefficient $n_d(a, b, c)$ has the meaning of the number of holomorphic maps $\mathbb{C}P^1 \to Y$ representing the homology class d and sending $0, 1, \infty \in \mathbb{C}P^1$ to given cycles in Y Poincare-dual to the classes $a, b, c \in H^*(Y)$ respectively. We refer to [5, 14] for variants of actual definitions which employ *intersection theory* in Kontsevich's compactifications of moduli spaces of stable maps $\mathbb{C}P^1 \to Y$ (see Section 1). In particular $n_d(a, b, c) = 0$ unless the total degree of the classes a, b, c equals the real dimension of the fundamental class in the space of maps $\mathbb{C}P^1 \to Y$ representing the class d. This gives rise to the grading $\deg q^d = 2 \int_{[d]} c(\mathcal{T}_Y)$ in the quantum cohomology algebra.

In addition to the associativity identity the structural constants of the quantum multiplication satisfy some integrability condition which can be formulated as compatibility of certain system of linear differential equations. Let $(t_1, ..., t_k)$ denote coordinated on $H^2(Y)$ with respect to a basis $(p_1, ..., p_k)$ of integral symplectic classes. The basis q^d in the semigroup ring $\mathbb{Z}[\Lambda]$ can be identified with $q_1^{d_1}...q_k^{d_k} = \exp(t_1 d_1 + ... + t_k d_k)$ where $(d_1, ..., d_k)$ are coordinates of $d \in H_2(Y)$ with respect to the dual basis. Denote t_0 the coordinate on $H^0(Y)$. The operators $p_i \circ$ of quantum multiplication act on vector functions $s(t_0, t)$ with values in $H^*(Y, \mathbb{C})$. In these notations the integrability condition reads:

$$\hbar \frac{\partial}{\partial t_0} s = s, \ \ \hbar \frac{\partial}{\partial t_i} s = p_i \circ s, i = 1, ..., k,$$

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form a consistent system of linear PDEs.

Solutions s to the PDE system can also be interpreted in terms of intersection theory on spaces of stable maps (see Section 1). The $\mathcal{D}$-module corresponding to this system is generated by a single formal vector-function $J_Y(t, \hbar^{-1})$ with coefficients in $H^*(Y, \mathbb{Q})$ (see Section 1). It has the following property [10]: whenever a scalar differential operator $D(\hbar\partial/\partial t, \exp t, \hbar)$ with coefficients in $\mathbb{C}[\hbar][[\Lambda]]$ annihilates (all components of the vector-function) J_Y, the symbol $D(p\circ, q, 0)$ vanishes in the quantum cohomology algebra of Y.

A construction of J_Y in terms of intersection theory on moduli spaces of stable maps is given in Section 1. By the very definition the function J_Y has the the asymptotical expansion

$$J_Y = e^{(t_0 + p_1 t_1 + ... + p_k t_k)/\hbar}(1 + o(1/\hbar)).$$

In applications to a toric complete intersection $Y \subset X$ we will detect degrees of holomorphic curves in Y by their degrees in X. This means that Λ will denote the semigroup of classes of compact holomorphic curves in X, $t_1, ..., t_k$ will be coordinates on $H_2(X)$, and J_Y — a formal series in $q_i = e^{t_i}$. Notice that in the case where $v_1, ..., v_l$ are positive and $\dim Y > 2$, the map $H_2(Y) \to H_2(X)$ is an isomorphism and thus the notation J_Y has the same meaning as before. However for semi-positive $v_1, ..., v_l$ (and $\dim Y > 1$) the map $H_2(Y) \to H_2(X)$ can have a non-trivial kernel, and therefore J_Y is obtained from the vector-function discussed in the previous paragraphs by the restriction operation. [1]

We will denote J the orthogonal projection of the vector-function J_Y to the subalgebra $H^*(\mathcal{V}, \mathbb{Q}) \subset H^*(Y, \mathbb{Q})$.

Introduce the formal vector-function $I(t, \hbar^{-1})$ with values in $H^*(\mathcal{V}, \mathbb{Q})$:

$$(*) \quad I = e^{(t_0 + p_1 t_1 + ... + p_k t_k)/\hbar} \sum_{d \in \Lambda} e^{t_1 d_1 + ... + t_k d_k} \times$$

$$\times \frac{\Pi_{a=1}^{l} \Pi_{m=-\infty}^{L_a(d)}(v_a + m\hbar) \; \Pi_{j=1}^{N} \Pi_{m=-\infty}^{0}(u_j + m\hbar)}{\Pi_{a=1}^{l} \Pi_{m=-\infty}^{0}(v_a + m\hbar) \; \Pi_{j=1}^{N} \Pi_{m=-\infty}^{D_j(d)}(u_j + m\hbar)},$$

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[1]I would like to thank V. Batyrev and B. Kim who pointed me to this subtlety in the semi-positive case.

where $D_j(d) = \int_{[d]} u_j$, $L_a(d) = \int_{[d]} v_a$. The function I is the product of $e^{(t_0+p\log q)/\hbar}$ with a power series in $q = e^t$ supported in Λ and weighted-homogeneous of degree 0 with respect to the grading

$$\deg \hbar = \deg u_j = \deg v_a = 1, \deg q^d = \int_{[d]} c(\mathcal{T}_Y).$$

The function J has the same properties (see Section 1).

Theorem 0.1. *Let Y be a non-singular toric complete intersection with the non-negative 1-st Chern class $c(\mathcal{T}_Y)$. Then the formal vector-functions I and J with coefficients in $H^*(\mathcal{V}, \mathbb{Q})$ coincide up to a triangular weighted-homogeneous change of variables:*

$$t_0 \mapsto t_0 + f_0(q)\hbar + h(q), \ \log q_i \mapsto \log q_i + f_i(q)$$

where $h, f_0, f_1, ..., f_k$ are weighted-homogeneous power series supported in $\Lambda - 0$ and $\deg f_0 = \deg f_i = 0$.

Remarks. 1) Comparisson of the asymptotic expansions for the components in $H^0(\mathcal{V}, \mathbb{Q})$ and $H^2(\mathcal{V}, \mathbb{Q})$ of the functions I and J in orders h^0 and h^{-1} uniquely determines the change of variables that transforms I to J (see Section 7).

2) The assumption $c(\mathcal{T}_Y) \geq 0$ guarantees that $\deg q^d \geq 0$ for all $d \in \Lambda - 0$. If $c(\mathcal{T}_Y)$ is positive, then the change of coordinates transforming I to J reduces to the multiplication by $\exp(h(q)/\hbar)$ where $h(q)$ is a polynomial supported at $\{d \in \Lambda | \int_{[d]} c(\mathcal{T}_Y) = 1\}$. Thus the functions I and J coincide whenever this support is empty (for instance, if $c(\mathcal{T}_Y)$ is a multiple of a positive integer class).

3) In the opposite case $\sum v_a = \sum u_j$ the manifold Y is a *Calabi-Yau* toric complete intersection (in the broad sense that $c(\mathcal{T}_Y) = 0$). Then $\deg q_i = 0$ for all i, $h(q) = 0$ and

$$\exp f_0 = \sum_{d:D_j(d)\geq 0} q^d \frac{L_1(d)!...L_l(d)!}{D_1(d)!...D_N(d)!}.$$

This and other components of the vector-function I are generalized hypergeometric functions.

Example: toric manifolds themselves. Taking $l = 0$ in Theorem 0.1 we arrive at the case $Y = X$ of Gromov – Witten theory on non-singular toric symplectic manifolds with $c(\mathcal{T}_X) \geq 0$. We describe further simplifications which occur in this case.

(a) If the anti-canonical class of the toric manifold is positive then $I = J_X$. The proof consists in elementary verification of the asymptotic expansion $I = \exp((t_0 + p \log q)/\hbar)\,(1 + o(1/\hbar))$. In particular, symbols of differential operators annihilating I yield the relations

$$u_1^{D_1(d)} ... u_N^{D_N(d)} = q^d$$

in the quantum cohomology algebra of X (see Corollary 0.4 below). This result agrees with the multiplicative structure in Floer cohomology of the loop space LX constructed in [7] by the method of generating functions (and based on discretization of loops) and with the results in [3, 16] on quantum cohomology of positive toric manifolds.

(b) If the anti-canonical class of the toric manifold is semi-positive then

$$I = e^{(t_0 + p \log q)/\hbar}\,[1 + p_1 f_1(q)/\hbar + ... + p_k f_k(q)/\hbar \,+\, o(1/\hbar)]$$

and differs from J_X only by the change of variables $\log q_i \mapsto \log q_i + f_i(q)$. We illustrate this case by the following instructive example which also exhibits *symplectic* invariance of Gromov – Witten theory.

Let X_1 and X_2 be the toric 3-folds obtained by projectivization of the following 3-dimensional vector bundles over $\mathbb{C}P^1$: $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$ for X_1 and $\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}$ for X_2. The bundles are topologically (and symplecticly) equivalent and the manifolds X_1 and X_2 are symplectomorphic. They are not isomorphic however as complex manifolds, and we shall see how the same Gromov – Witten invariants emerge from different computational procedures.

The manifolds X_1, X_2 are obtained by factorization of $\mathbb{C}^5$ by the torus $T_{\mathbb{C}}^2$ embedded into the 5-torus of diagonal matrices as prescribed by the matrices M_1^t and M_2^t respectively:

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The columns of the matrix M represent the classes $u_1, ..., u_5$ as linear combinations of a basis p_1, p_2 in $H^2(X)$: for X_1 we have $u_1 = u_2 = p_1, u_3 = p_2, u_4 =$

$u_5 = p_2 - p_1$. Using upper-case notations for X_2 we get $U_1 = U_2 = p_1, U_3 = U_4 = p_2, U_5 = p_2 - 2p_1$. We refer the reader to Section 3 for a detailed combinatorial description of cohomological invariants of toric manifolds. Using that description we find the multiplicative relations $u_1 u_2 = 0, u_3 u_4 u_5 = 0$ and $U_1 U_2 = 0, U_3 U_4 U_5 = 0$ in the cohomology algebra and observe that they are effectively the same:

$$H^*(X) = \mathbb{Z}[p_1, p_2]/(p_1^2, p_2^3 - 2p_2^2 p_1).$$

The Kähler cone in both cases consists of $p_1 t_1 + p_2 t_2$ with $t_1, t_2 > 0$. The anti-canonical class $\sum u_j = \sum U_j = 3p_2$, and the semigroup algebra $\mathbb{Q}[[\Lambda]]$ of identifies with $\mathbb{Q}[[q_1, q_2]]$ with the grading $\deg q_1 = 0, \deg q_2 = 3$.

The series I corresponding to X_1 has the form

$$I_1 = e^{(t_0 + p_1 \log q_1 + p_2 \log q_2)/\hbar} \times$$

$$\sum_{d_1, d_2 = 0}^{\infty} \frac{q_1^{d_1} q_2^{d_2} \ \Pi_{m=-\infty}^0 (p_2 - p_1 + m\hbar)^2}{\Pi_{m=1}^{d_1}(p_1 + m\hbar)^2 \ \Pi_{m=1}^{d_2}(p_2 + m\hbar) \ \Pi_{m=-\infty}^{d_2 - d_1}(p_2 - p_1 + m\hbar)^2}$$

It has the asymptotics $\exp((t_0 + p \log q)/\hbar) \ [1 + o(1/\hbar)]$ and thus coincides with J_X.

The series I_1 is annihilated by the differential operators

$$\Delta_1 = (\hbar \frac{\partial}{\partial \log q_1})^2 - q_1 (\hbar \frac{\partial}{\partial \log q_2} - \hbar \frac{\partial}{\partial \log q_1})^2,$$

$$\Delta_2 = \hbar \frac{\partial}{\partial \log q_2} (\hbar \frac{\partial}{\partial \log q_2} - \hbar \frac{\partial}{\partial \log q_1})^2 - q_2.$$

Thus the relations

$$p_1^2 = q_1 (p_2 - p_1)^2, \quad p_2 (p_2 - p_1)^2 = q_2$$

describe the quantum deformation of the cohomology algebra of X.

The series I corresponding to X_2 has the form (we use the upper-case notations Q_1, Q_2 instead of q_1, q_2):

$$I_2 = e^{(t_0 + p_1 \log Q_1 + p_2 \log Q_2)/\hbar} \times$$

$$\sum_{d_1, d_2 = 0}^{\infty} \frac{Q_1^{d_1} Q_2^{d_2} \ \Pi_{m=-\infty}^0 (p_2 - 2p_1 + m\hbar)}{\Pi_{m=1}^{d_1}(p_1 + m\hbar)^2 \ \Pi_{m=1}^{d_2}(p_2 + m\hbar)^2 \ \Pi_{m=-\infty}^{d_2 - 2d_1}(p_2 - 2p_1 + m\hbar)}.$$

It has the asymptotics $\exp((t_0 + p \log Q)/\hbar)\ [1 + (2p_1 - p_2)f(Q_1)/\hbar + o(1/\hbar)]$ where

$$f(Q_1) = \sum_{d_1=1}^{\infty} \frac{(2d_1 - 1)!}{(d_1!)^2} Q_1^{d_1}.$$

We have to put therefore $q_1 = Q_1 \exp(2f(Q_1))$, $q_2 = Q_2 \exp(-f(Q_1))$.

In fact the inverse change of variables is given by the simple formulas

$$Q_1 = \frac{q_1}{(1 + q_1)^2}, \ Q_2 = q_2(1 + q_1),$$

$$Q_2 \frac{\partial}{\partial Q_2} = q_2 \frac{\partial}{\partial q_2}, \ Q_2 \frac{\partial}{\partial Q_2} - 2Q_1 \frac{\partial}{\partial Q_1} = \frac{1 + q_1}{1 - q_1}(q_2 \frac{\partial}{\partial q_2} - 2q_1 \frac{\partial}{\partial q_1}).$$

With these formulas at hands it is straightforward to check that the change of variables transforms the system of differential equations

$$\hbar^3 (Q_2 \frac{\partial}{\partial Q_2})^2 (Q_2 \frac{\partial}{\partial Q_2} - 2Q_1 \frac{\partial}{\partial Q_1})\ I = Q_2\ I$$

$$(Q_1 \frac{\partial}{\partial Q_1})^2\ I = Q_1(Q_2 \frac{\partial}{\partial Q_2} - 2Q_1 \frac{\partial}{\partial Q_1})(Q_2 \frac{\partial}{\partial Q_2} - 2Q_1 \frac{\partial}{\partial Q_1} - 1)\ I$$

satisfied by $I = I_2$ to the system $\Delta_1 I = 0, \Delta_2 I = 0$ satisfied by $I = I_1$. This guarantees that $I_2(Q(q)) = I_1(q)$ and finally gives rise to the same description of the quantum cohomology algebra. $\square$

Our proof of Theorem 0.1 is based on an equivariant generalization of the Gromov – Witten theory.

Let $\mathcal{V}$ be a holomorphic l - dimensional vector bundle over the n - dimensional Kähler manifold X equivariant with repect to a hamiltonian Killing action of a torus G. It was explained in [10] how to extend the Gromov – Witten theory to the G - super - manifolds $(X, \mathcal{V})$ of dimension (n, l) in the case of *convex* $\mathcal{V}$ (i.e. bundles with all fibers spanned by global holomorphic sections). [2]

The equivariant quantum cohomology algebra of the super - manifold $(X, \mathcal{V})$ is a deformation of the cup - product in the equivariant cohomology algebra $H_G^*(X, \mathbb{Q})$ provided with the Poincaré pairing $\langle f, g \rangle = \int_{[X]} fg\ Euler(\mathcal{V})$

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[2] The idea of the construction is due to M. Kontsevich (see [12, 10]) while the terminology of super-manifolds in this context was introduced by A. Schwarz [17].

with values in $H_G^*(point) = H^*(BG, \mathbb{Q})$. Here the equivariant top Chern class $Euler(\mathcal{V})$ is assumed to be invertible over the field of fractions of the polynomial algebra $H^*(BG, \mathbb{Q}) \simeq \mathbb{Q}[\lambda_1, ..., \lambda_{\dim G}]$. Respectively, the equivariant quantum cohomology algebra and the quantum cohomology $\mathcal{D}$-module of the super-manifold are defined over this field of fractions. It is important however that the structural constants $\langle a \circ b, c \rangle$ and the components $\langle J_\mathcal{V}, a \rangle$ of the corresponding solution vector-function are defined over the polynomial algebra $\mathbb{C}[\lambda]$ (as some equivariant Poincaré pairings with the top Chern classes of suitable vector bundles over the moduli spaces of stable maps to X, see Section 1). In the non-equivariant limit to $\lambda = 0$ the algebra $(H_G^*(X)/ker\langle \cdot, \cdot \rangle)$ degenerates to $H^*(\mathcal{V})$, and the corresponding structural constants and solutions turn into their counterparts in the Gromov-Witten theory on the complete intersection $Y \subset X$ defined by a generic holomorphic section of $\mathcal{V}$. This allows to obtain Theorem 0.1 as the specialization to $\lambda = 0$ of the following result about toric super-manifolds.

Let us consider the equivariant cohomology algebra $H_{T^N}^*(X)$ of the toric manifold $X = \mathbb{C}^N//T^k$ provided with the action of the torus T^N of diagonal unitary matrices. The coefficient algebra $H^*(BT^N)$ of the equivariant theory is canonically isomorphic to the algebra $\mathbb{Z}[\lambda_1, ..., \lambda_N]$ of polynomial functions on $LieT^N$. The algebra $H_{T^N}^*(X)$ is multiplicatively generated over $\mathbb{Z}[\lambda]$ by the equivariant counterpart $p_1, ..., p_k \in H^2(X)$ of a basis in $H^2(X)$. The equivariant classes $u_1, ..., u_N \in H_G^2(X)$ Poincare-dual to the G-invariant toric hypersurfaces corresponding to the coordinate hyperplanes in $\mathbb{C}^N$ are some linear combinations

$$u_j = \sum_{i=1}^k p_i m_{ij} - \lambda_j, \ j = 1, ..., N,$$

of these generators. The relations between the generators p_i can be written in the form (see Section 3):

$$u_{j_1}...u_{j_r} = 0, \ j_1 < ... < j_r,$$

whenever the toric hypersurfaces corresponding to $u_{j_1}, ..., u_{j_r}$ have empty intersection.

Let $\mathcal{V}$ denote the direct sum of $l \geq 0$ non-negative T^N-equivariant line

bundles over X with the *non-zero* equivariant 1-st Chern classes

$$v_a = \sum_{i=1}^{k} p_i l_{ia} - \lambda'_a, \ a = 1, ..., l.$$

The variables $(\lambda', ..., \lambda'_l)$ here stand for the generators of $H^*(BT^l)$ where the torus T^l acts fiberwise on the bundle $\mathcal{V}$. (One may think of the toric super-manifold $(X, \mathcal{V})$ as of the symplectic reduction of the super-space $(\mathbb{C}^N, \mathbb{C}^l)$ by the torus T^k embedded into the maximal torus $G = T^N \times T^l$ by means of the matrix $(m_{ij}|l_{ia})$.)

Denote $J_{\mathcal{V}}(t_0, \log q, \hbar^{-1})$ the solution formal vector-function of the equivariant Gromov – Witten theory on the super-manifold $(X, \mathcal{V})$. Define another formal vector-function $I_{\mathcal{V}}(t_0, \log q, \hbar^{-1})$ by the formal series $(*)$. Coefficients of both formal functions are G-equivariant cohomology classes of X over $\mathbb{Q}[\lambda, \lambda']$. In the limit $(\lambda, \lambda') = 0$ the functions $J_{\mathcal{V}}$ and $I_{\mathcal{V}}$ yield J and I respectively.

Theorem 0.2. *Suppose that* 1-*st Chern class* $\sum u_j - \sum v_a$ *of the super-manifold* $(X, \mathcal{V})$ *is non-negative. Then* $J_{\mathcal{V}}$ *coincides with* $I_{\mathcal{V}}$ *up to a weighted-homogeneous triangular change of variables:*

$$t_0 \mapsto t_0 + f_0(q)\hbar + \sum \lambda_j g_j(q) + h(q), \ \log q_i \mapsto \log q_i + f_i(q), \ i = 1, ..., k$$

where f_0, f_i, g_j, h *are weighted-homogeneous formal* q-*series with* $\deg f_0 = \deg f_i = \deg g_j = 0, \deg h = 1$ *supported at* $\Lambda - 0$.

Corollary 0.3. *Suppose that a linear differential operator* $D(\hbar\partial/\partial \log q, q, \lambda, \hbar)$ *with coefficients in* $\mathbb{C}[\lambda, \hbar][[\Lambda]]$ *annihilates the vector-function* $I_{\mathcal{V}}$ *transformed to the new variables. Then the relation* $D(p\circ, q, \lambda, 0)$ $= 0$ *holds in the quantum cohomology algebra of the super-manifold* $(X, \mathcal{V})$. *In particular, in the quantum cohomology algebra of the complete intersection* $Y \subset X$ *(with* $\dim Y > 1$) *we have* $\langle a, D(p\circ, q, 0, 0) \circ b \rangle = 0$ *for any* $a, b \in H^*(\mathcal{V}) \subset H^*(Y)$.

Remarks. 4) The change of variables transforming $I_{\mathcal{V}}$ to $J_{\mathcal{V}}$ is uniquely determined by the asymptotics of $I_{\mathcal{V}}$ modulo $\hbar^{-2}$.

5) The q^d-term in $\langle a, b \circ p_{i_1} \circ ... \circ p_{i_r} \rangle$ has the following enumerative meaning: it is the (virtual) number of degree d holomorphic maps $\mathbb{C}P^1 \to Y$ which send a given generic configuration of $r + 2$ distinct points in $\mathbb{C}P^1$ to the

given generic cycles a, b and r given generic hypersurfaces Poincare-dual to $p_{i_1}, ..., p_{i_r}$ respectively.

6) Due to a somewhat subtle relationship between the quantum cohomology algebras of Y and $(X, \mathcal{V})$ it seems to be dangerous to deduce enumerative corollaries about Y directly from differential equations for J (instead of $J_{\mathcal{V}}$ or J_Y). We do not know counter-examples however. In many cases such corollaries can be justified by means of additional dimensional or Hodge-theoretic arguments. In particular, if $\oplus H^{r,r}(Y) \subset H^*(\mathcal{V}, \mathbb{C})$ (for example if $Y \subset X$ is a hypersurface of odd dimension), we have $\langle a, J_Y \rangle = \langle a, J \rangle$ for all $a \in H^*(Y)$. In this case $D(p\circ, q, 0) = 0$ if the differential operator $D(\hbar\partial/\partial \log q, q, \hbar)$ annihilates I transformed to the new variables as described in Theorem 0.1.

Consider now the differential equations satisfied by $I_{\mathcal{V}}$. Put

$$\partial_j = \sum_i m_{ij} \hbar \partial/\partial \log q_i - \lambda_j, \ j = 1, ..., N,$$

$$\partial_a' = \sum_i l_{ia} \partial/\partial \log q_i - \lambda_a', \ a = 1, ..., l.$$

For each $d \in \Lambda$ with $D_1(d), ..., D_N(d) \geq 0$ we introduce the differential operator (see [9]):

$$\Delta_d = \Pi_j \Pi_{m=0}^{D_j(d)-1}(\partial_j - m\hbar) - q^d \Pi_a \Pi_{m=1}^{L_a(d)}(\partial_a' + m\hbar).$$

It follows easily from the detailed description of the equivariant cohomology algebra $H_G^*(X)$ given in Section 3 that $\Delta_d I_{\mathcal{V}} = 0$. (In fact $I_{\mathcal{V}}$ may satisfy some stronger differential equations.)

Denote $\hat{\Delta}_d$ the polynomial

$$u_1^{D_1(d)}...u_N^{D_N(d)} - q^d v_1^{L_1(d)}...v_l^{L_l(d)}$$

in the quantum cohomology algebra of the non-negative toric complete intersection $Y \subset X$.

Corollary 0.4. *Suppose that for the toric complete intersection $Y \subset X$ we have $J_{\mathcal{V}} = I_{\mathcal{V}}$ (for instance due to the causes described in Remark 2). Then $\langle a, b \circ \hat{\Delta}_d \rangle = 0$ for any $a, b \in H^*(\mathcal{V}) \subset H^*(Y)$.*

We would like to emphasise the hypergeometric character of the function $I_{\mathcal{V}}$. It is easy to see that the differential operators Δ_d annihilate also the

following hypergeometric integrals:

$$\int_{\Gamma \subset E_q} e^{(\sum u_j - \sum v_a)/\hbar} \, u_1^{\lambda_1/\hbar}...u_N^{\lambda_N/\hbar} \, v_1^{\lambda_1'/\hbar}...v_l^{\lambda_l'/\hbar} \times$$

$$\times \, \frac{d\log u_1 \wedge ... \wedge d\log u_N \wedge dv_1 \wedge ... \wedge dv_l}{d\log q_1 \wedge ... \wedge d\log q_k} \,.$$

Here Γ_q is a suitable real $N - k + l$-dimensional (non-compact) cycle in the complex $N - k + l$-dimensional variety

$$E_q = \{(u,v)|\Pi_{j=1}^N u_j^{m_{ij}} = q_i \Pi_{a=1}^l v_a^{l_{ia}}, i = 1,...,k\}$$

provided with the local coefficient system $u^{\lambda/\hbar} v^{\lambda'/\hbar}$.

Example: a mirror theorem. Partition the variables $u_1, ..., u_N$ into $l + 1$ groups and denote F_a the sum of u_j in each group, $a = 0, ..., l$. Define the matrix (l_{ia}) in such a way that $v_a = F_a(u), a = 1, ..., l$, correspond to convex line bundles on X. In the limit $\lambda' = 0, \lambda = 0$ the above integral evaluated explicitly over $\mathbb{R}_+^l$ in dv reduces to

$$\int_{\Gamma_q' \subset X_q'} e^{F_0(u)/\hbar} \frac{d\log u_1 \wedge ... \wedge d\log u_N}{(1 - F_1(u))...(1 - F_l(u)) \, d\log q_1 \wedge ... \wedge d\log q_k}$$

where $X_q' = \{u|\Pi_j u_j^{m_{ij}} = q_i, \, i = 1, ..., k\}$. Further reduction by Cauchy residue formula yields

$$(**) \quad \int_{\gamma_q \subset Y_q'} e^{F_0(u)/\hbar} \frac{d\log u_1 \wedge ... \wedge d\log u_N}{dF_1 \wedge ... \wedge dF_l \wedge d\log q_1 \wedge ... \wedge d\log q_k}.$$

Here γ_q can be understood as Morse-theoretic cycles of the function $\operatorname{Re} F_0$ restricted to the $N - k - l$-dimensional manifold

$$Y_q' = \{u|\Pi_j u_j^{m_{ij}} = q_i, i = 1, ..., l, \, F_a(u) = 1, a = 1, ..., l\} \,.$$

It is not hard to see that all components of the vector-function I are described by such integrals with suitable γ_q. [3] This constitutes the content of the mirror symmetry between the toric complete intersections Y and Y'.

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[3] In general the converse is not true — different toric manifolds $X = \mathbb{C}^N//T^k$ obtained by the reduction on different generic levels of the momentum map have different cohomology algebras $H^*(\mathcal{V})$ and may give rise to different I whose integral representations differ by the choice of cycles only.

Indeed:

(a) In the case of a Calabi–Yau toric complete intersection Y we have $\sum u_j = \sum v_a$ and thus $F_0(u) = 0$. The affine varieties X_q' in this case can be compactified (see [2]) to the toric varietiy $\hat{X}'$ with the momentum polyhedron *polar* to that for X. The varieties Y_q' are compactified to (singular) Calabi–Yau complete intersections $\hat{Y}_q' \subset \hat{X}'$. The forms $d \log u/dv \wedge d \log q$ extend to holomorphic volume forms ω_q on desingularizations of $\hat{Y}_q'$. Thus the components of the hypergeometric series I are identified with the periods $\int_{\gamma_{\log q}} \omega_q$ responsible for variations of complex structures in a family Calabi–Yau manifolds birationally isomorphic to $\hat{Y}_q'$.

According to V.Batyrev [2] the q-family of Calabi–Yau manifolds is *mirror - symmetric* to the original family of toric complete intersections Y, and thus our Theorem 0.1 confirms the *mirror conjecture* for Calabi – Yau toric complete intersections formulated in detail in [4].

(b) In the lecture [9] we suggested a generalisation of the mirror conjecture beyond the class of Calabi – Yau manifolds. [4] In this generalization the quantum cohomology $\mathcal{D}$-module of a compact Kähler manifold Y should be equivalent, up to some change of variables, to the $\mathcal{D}$-module generated by the oscillating integrals $\int e^{f_q/\hbar} \omega_q$ defined by a suitable family (Y_q', ω_q, f_q) of (possibly non-compact) complex algebraic manifolds Y_q' (of the same dimension as Y) provided with holomorphic volume forms ω_q and holomorphic phase functions f_q. This generalisation was confirmed for toric Fano manifolds in [9], for Fano and Calabi–Yau projective complete intersections in [10] and for flag manifolds of the series A in [11]. Thus Theorem 0.1 along with the integral representation $(**)$ proves our generalized mirror conjecture for non-negative toric complete intersections described at the beginning of this example. $\square$

The remaining part of the paper contains a proof of Theorem 0.2. It exploits some general properties of equivariant Gromov – Witten invariants of super-manifolds $(X, \mathcal{V})$ described in [10] in the setting of *convex* Kähler manifolds X. Foundations of the Gromov – Witten theory for general X were developed recently by several groups of authors (see for instance [14]). It appears that these results admit straightforward equivariant generalizations.

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[4]A similar generalization was recently proposed in [6].

However there is no ready reference for such a generalization in the literature. In Section 1 we describe the properties of the equivariant Gromov–Witten theory we use in this paper in the axiomatic form. These properties have been verified in [10] for convex X. Thus the proof of Theorems 0.1 and 0.2 given in this paper is complete in the case of non-negative complete intersections in products of projective spaces (such products are convex), and in the case of more general toric manifolds still should be complemented by a verification of the axioms.

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1. Moduli spaces of stable maps. Let (C, x) be a compact connected complex algebraic curve with at most double singular points and with r pairwise distinct non-singular marked points $x = (x_1, ..., x_r)$. We will assume that C has the arithmetic genus $\dim H^1(C, \mathcal{O}) = 0$. Two holomorphic maps $(C, x) \to M, (C', x') \to M$ of two such curves to a complex manifold Y are called *equivalent* if they are identified by a holomorphic isomorphism $(C, x) \to (C', x')$. A holomorphic map $(C, x) \to M$ is called *stable* (see [12]) if it does not admit non-trivial infinitesimal automorphisms. A stable map may have a non-trivial finite group of discrete automorphisms.

The *degree* of a holomorphic map $(C, x) \to M$ is defined as the homology class $d \in H_2(M)$ it represents.

Denote $M_{r,d}$ the set of equivalence classes of degree d genus 0 stable maps $(C, x) \to M$ with r marked points. For a compact projective variety M the set $M_{r,d}$ has a natural structure of a compact complex algebraic variety (see [12, 5]). We will call the spaces $M_{r,d}$ the *moduli spaces* of stable maps.

If M is a homogeneous Kähler space of a compact semi-simple Lie group, the moduli spaces $M_{n,d}$ are known [5] to be *non-singular orbifolds* i.e. local quotients of non-singular spaces by finite groups. In particular the spaces have natural fundamental cycles providing Poincaré duality over $\mathbb{Q}$. For general M it is still convenient to think of the moduli spaces $M_{n,d}$ as of *singular orbifolds* i.e. local quotients of singular spaces by finite groups. The finite groups in question are the discrete automorphism groups of stable maps, and many local constructions in the moduli spaces become transparent only after passing to their local "unquotient" coverings. Vector bundles and their char-

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acteristic classes provide important examples of the orbifold ideology. By a vector bundle over $M_{n,d}$ we mean a sheaf which is locally identified with the sheaf of invariant sections of a natural local vector bundle on such coverings. The structural group G of the bundle acts with at most finite stabilizers on the total space of corresponding principal bundle $P \to M_{n,d}$. In this case the rational coefficient equivariant cohomology algebra $H_G^*(P, \mathbb{Q})$ naturally identifies with $H^*(M_{n,d}, \mathbb{Q})$. Then the augmentation homomorphism $H^*(BG, \mathbb{Q}) \to H_G^*(P, \mathbb{Q})$ induced by the G-equivariant map $P \to pt$ defines characteristic classes of the bundle. Talking about vector bundles over the moduli spaces, their Euler and Chern classes we will always have this orbifold subtlety in mind.

According to [15, 14] the moduli spaces $M_{n,d}$ can be provided with virtual fundamental cycles — rational homology classes of $M_{r,d}$ satisfying the axioms [13] of Gromov - Witten theory. We call the virtual fundamental cycles *Gromov - Witten classes.* Both the axioms and their realization by means of Gromov - Witten classes allow dependence of M on parameters.

Suppose that a compact Lie group G acts on a compact Kähler manifold M by hamiltonian automorphisms of the Kähler structure. Then G acts naturally on the Moduli spaces $M_{r,d}$. Let $B \subset BG$ be a finite-dimensional approximation to the classifying space BG of principal G-bundles, and $M_B \to B$ — the associated M-bundle. The bundle can be considered as a family of compact Kähler manifolds. This allows one to construct parametric Gromov - Witten classes in the moduli spaces $(M_{r,d})_B \to B$. Exhausting BG by the finite-dimensional approximations one can construct *G-equivariant* Gromov-Witten classes $[M_{r,d}]$. The recent progress [15, 14] in foundations of Gromov – Witten theory leaves little doubt that the "virtual fundamental class" approach is consistent with the axioms [13, 10] of *equivariant* Gromov – Witten theory.

We describe below a variant of the axioms in the form convenient for applications in the present paper.

(1) The equivariant Gromov - Witten class defines an $H^*(BG, \mathbb{Q})$ - linear function $\int_{[M_{r,d}]} : H_G^*(M_{r,d}, \mathbb{Q}) \to H^*(BG, \mathbb{Q})$ of homogeneity degree $-2[(c_1(\mathcal{T}_M), d) + \dim_{\mathbb{C}} M + r - 3]$. The moduli spaces $M_{r,0}$ are isomorphic to $M \times \bar{\mathfrak{M}}_r$ where $\bar{\mathfrak{M}}_r$ is the Deligne - Mumford compactification of the moduli space $\mathfrak{M}_r$ of ordered r-tuples of distinct points on $\mathbb{C}P^1$. The Gromov - Witten classes $[M_{r,0}]$ are (equivariant) fundamental cycles of the manifolds $M_{r,0}$

for $r \geq 3$ and are not defined for $r = 0, 1, 2$.

Remark. It is crucial for applications in this paper that the equivariant Gromov – Witten class is indeed defined over the polynomial algebra $H^*(BG, \mathbb{Q})$ and not over its field of fractions.

(2) There exist natural equivariant *forgetting maps* $\mathrm{fgt}_s : M_{r+1,d} \to M_{r,d}$ defined (see [12, 5]) by forgetting the marked point x_s, $1 \leq s \leq r+1$. The fiber of fgt_s over the point represented by a stable map $(C, x) \to M$ is canonically isomorphic to the quotient of C by the finite group of automorphisms of the stable map.

(3) There exist natural equivariant *evaluation maps* $\mathrm{ev}_s : M_{r,d} \to M$ defined by evaluation of a stable map at the marked point x_s. The 3-*point correlators*

$$\langle a, b, c \rangle := \sum_d q^d \int_{[M_{3,d}]} \mathrm{ev}_1^*(a) \, \mathrm{ev}_2^*(b) \, \mathrm{ev}_3^*(c)$$

are structural constants $\langle a \circ b, c \rangle$ of a (super) commutative associative Frobenius algebra structure on $(H_G^*(M), \langle \cdot, \cdot \rangle)$ with the unity 1. Here q^d stands for the element in the group ring of the lattice $H_2(M)$ corresponding to $d \in H_2(M)$, and the (super) symmetric bilinear form $\langle a, b \rangle$ is the (equivariant) Poincare pairing $\int_M ab$. The axiom means that the new $H^*(BG, \mathbb{Q}[[q]])$-bilinear multiplication $\circ$ on $H_G^*(M, \mathbb{Q}[[q]])$ defined by

$$\langle a \circ b, c \rangle := \langle a, b, c \rangle \ \forall a, b, c \in H_G^*(M)$$

is associative, and that the class $1 \in H_G^0(M)$ plays the role of the unity:

$$\langle a, b, 1 \rangle = \langle a, b \rangle \ \forall a, b \in H_G^*(M).$$

Here $\mathbb{Q}[[q]]$ is the formally completed semigroup algebra of the semigroup of degrees $d \in H_2(M)$ of compact holomorphic curves in M. The algebra $(H_G^*(M, \mathbb{Q}[[q]]), \ \circ)$ is called *the (equivariant) quantum cohomology algebra of M*. Reduced modulo the maximal ideal in $\mathbb{Q}[[q]]$, it identifies with the "classical" cohomology algebra $H_G^*(M, \mathbb{Q})$ since $M_{3,0} = M$.

(4) Let $d = (d_1, ..., d_k)$ denote coordinates of d with respect to a basis in $H_2(M)$, and $p_1, ..., p_k \in H_G^2(M)$ represent the dual basis in $H^2(M)/H^2(BG)$. The $H^*(BG)$ - linear operators $p_i \circ$ of quantum multiplication by p_i satisfy

$$q_i \partial(p_j \circ)/\partial q_i = q_j \partial(p_i \circ)\partial q_j.$$

These identities along with associativity and commutativity of the quantum multiplication mean that the following linear system of parial differential equations is consitent for any value of the parameter $\hbar \neq 0$:

$$\hbar \frac{\partial}{\partial t_0} s = s, \quad \hbar q_i \frac{\partial}{\partial q_i} s = p_i \circ s, \quad i = 1, ..., k.$$

(5) When a runs an $H^*(BG)$-basis in $H_G^*(M)$, the following formal vector - functions s_a run a basis of solutions to the above PDE system (see e. g. [10]). Define the (equivariant) class s_a of M (with appropriate coefficients) by

$$\forall b \in H_G^*(M) \ \langle s_a, b \rangle = \int_M e^{(t_0 + p_1 \log q_1 + ... + p_k \log q_k)/\hbar} ab +$$

$$\sum_{d \neq 0} q^d \int_{[M_{2,d}]} \frac{\mathrm{ev}_1^*(e^{(t_0 + p_1 \log q_1 + ... + p_k \log q_k)/\hbar} a)}{\hbar - c} \ \mathrm{ev}_2^*(b).$$

Here c is the (equivariant) 1-st Chern class of the following line bundle over $M_{2,d}$ called *the universal cotangent line at the 1-st marked point*. The forgetful map $\mathrm{fgt}_3 : M_{3,d} \to M_{2,d}$ (defined for $d \neq 0$) has the section $\mathrm{mk}_1 : M_{2,d} \to M_{3,d}$ defined by marked point x_1 in each fiber. The universal cotangent line is defined as the conormal bundle to the hypersurface $\mathrm{mk}_1(M_{2,d})$ in $M_{3,d}$. The fiber of this bundle at the point represented by the stable map $(C, x_1, x_2) \to M$ is the cotangent line $T_{x_1}^* C$.

In particular, this axiom implies (see [10]) that whenever a differential operator

$$D(\hbar q_1 \partial/\partial q_1, ..., \hbar q_k \partial/\partial q_k, q_1, ..., q_k, \hbar)$$

with coefficients in $H^*(BG, \mathbb{C}[[q, \hbar]])$ annihilates simultaneously all functions $\langle s_a, 1 \rangle$, $a \in H_G^*(M)$, the relation $D(p_1 \circ, ..., p_k \circ, q_1, ..., q_k, 0) = 0$ defined by the symbol of this operator holds true in the quantum cohomology algebra of M. The vector-function J_M mentioned in the introduction is defined by $\forall a \ \langle s_a, 1 \rangle = \langle a, J_M \rangle$.

(6) A holomorphic vector bundle $\mathcal{V}$ over M is called *convex* if each fiber of $\mathcal{V}$ is generated by global holomorphic sections of $\mathcal{V}$. The Gromov – Witten theory of the *super-manifold* $(M, \mathcal{V})$ is constructed by taking $f \mapsto \int_{[M_{r,d}]} f \ Euler(\mathcal{V}_{n,d})$ on the role of Gromov – Witten classes. Here $\mathcal{V}_{n,d}$ is the vector bundle

$$\mathcal{V}_{r,d} = (\mathrm{fgt}_{r+1})_* (\mathrm{ev}_{r+1}^*(\mathcal{V})).$$

over the moduli space $M_{n,d}$. The fiber of $\mathcal{V}_{r,d}$ over the point in $M_{r,d}$ represented by the degree d stable map $\phi : (C,x) \to M$ is $H^0(C, \phi^*(\mathcal{V}))$ and has the complex dimension $(c_1(\mathcal{V}), d)$ since $H^1(C, \phi^*(\mathcal{V})) = 0$ due to the convexity of the bundle. Evaluation at a marked point defines an epimorphism $\mathcal{V}_{n,d} \to \mathcal{V}$ of the bundles over $\mathrm{ev}_s : M_{n,d} \to M$. This allows one to introduce the correlators $\langle a, b, c \rangle$, $\langle s_a, b \rangle$ for a, b, c from the quotient $H^*(\mathcal{V})$ of $H^*(X, \mathbb{Q})$ by the kernel of the intersection form $\langle a, b \rangle = \int_M a \ b \ Euler(\mathcal{V})$ and thus to define for the super-manifold $(X, \mathcal{V})$ the quantum cohomology algebra and the quantum cohomology $\mathcal{D}$-module satisfying the analogues of the axioms (3),(4),(5) of the super-manifold $(X, \mathcal{V})$. Analogous axioms hold true in the equivariant setting provided that the convex bundle $\mathcal{V}$ is G-equivariant. Then $Euler(\cdot)$ means the equivariant Euler class. In particular, the vector-function $J_{\mathcal{V}}$ from Introduction is the counterpart of J_M in the case of the (equivariant) super-manifolds $(X, \mathcal{V})$.

(7) Let Y be a non-singular submanifold in M defined as the zero locus of a section of the convex vector bundle $\mathcal{V}$. The inclusion $i : Y \subset M$ identifies the Frobenius algebra $(H^*(\mathcal{V}), a \mapsto \int_M a \ Euler(\mathcal{V}))$ with the image of the homomorphism $i^* : H^*(M, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$. Respectively, the inclusion of the moduli spaces $Y_{n,d} \to M_{n,i_*d}$ maps the Gromov – Witten class $[Y_{n,d}]$ to the Gromov – Witten class corresponding to the super-manifold $(X, \mathcal{V})$. This implies that the correlators $\langle a, b, c \rangle, \langle s_a, b \rangle$ for Y of the classes a, b, c induced from M can be computed by integration over the Gromov – Witten classes $[M_{n,d}]$ against the Euler classes of $\mathcal{V}_{n,d}$. In particular the orthogonal projection J of the vector-function J_Y to $i^*(H^*(M, \mathbb{Q}))$ identifies with (the non-equivariant version of) $J_{\mathcal{V}}$.

(8) The Borel fixed point localization technique applies to the Gromov - Witten classes $[M_{r,d}]$. Consider the action of the torus G on $M_{r,d}$. A fixed point of this action is represented by a map $\phi : (C,x) \to M$ such that each irreducible component of C is mapped either to the fixed point manifold M^G or onto a complex 1-dimensional orbit of $G_{\mathbb{C}}$. We will formulate the axiom in the special case where both the set of fixed points M^G and the set of 1-dimensional orbits are finite. If it is the case, each connected component of the fixed point set in $M_{r,d}$ is (the quotient by a finite group of) a product $\Pi \, \bar{\mathfrak{M}}_n$ of Deligne-Mumford configuration spaces $\bar{\mathfrak{M}}_n$. Each factor in this product parametrizes configurations of marked and singular points on a 1-dimensional connected component of $\phi^{-1}(M^G) \subset C$. On each of the remaining irreducible components of C the map ϕ covers a compactified 1-

dimensional orbit of G (with certain multiplicity in which case it ramifies at the two compactifying fixed points). We will call these components *edges* of the curve C.

The localization of the equivariant Gromov - Witten class $[M_{r,d}]$ at the fixed point component in $M_{r,d}$ is

$$f \mapsto \frac{1}{|Aut|} \int_{\Pi \, \bar{\mathfrak{M}}_n} \frac{f}{\text{Virtual Normal Euler Class}} \; .$$

Here Aut is the automorphism group of a typical stable map ϕ from the fixed point component in question.

In order to describe the Virtual Normal Euler Class consider the vector spaces

$$\mathcal{N}_i = H^i(C, \phi^*(\mathcal{T}_M)), i = 0, 1, \; \mathcal{N}_2 = H^0(C, \mathcal{T}_C[x]), \; \mathcal{N}_3 = \oplus_y T'_y \otimes T''_y \; ,$$

where y runs double points of C situated on the edges, and T'_y, T''_y are the tangent spaces to the two components of C intersecting at y. Dimensions of the spaces $\mathcal{N}_i$ do not change along the connected fixed point component of $M_{r,d}$ and thus they form vector bundles over $\Pi \, \bar{\mathfrak{M}}_n$. These bundles carry natural infinitesimal actions of G. Indeed, this is obvoius for $\mathcal{N}_0$ and $\mathcal{N}_1$. The differential ϕ_* identifies the space $\mathcal{N}_2$ of infinitesimal automorphisms of (C, x) with a G-invariant subspace in $\mathcal{N}_0$ due to stability of ϕ. For a double point y on the edge C' where ϕ is an m-fold covering map, the space $(T_y C)^{\otimes m}$ is identified with the tangent line to the closed 1-dimensional G-orbit at a fixed point and thus inherits an infinitesimal action of G. The Virtual Normal Euler Class is defined as the following combination of equivariant Euler classes of the bundles $\mathcal{N}_i$:

$$\frac{Euler(\mathcal{N}_0)Euler(\mathcal{N}_3)}{Euler(\mathcal{N}_1)Euler(\mathcal{N}_2)}.$$

Remark. One can use this formula in order to *define* the Gromov - Witten cycle in the equivariant theory. However the verification of the other axioms and especially — the *polynomiality* property (1) of the Gromov - Witten class to be defined over $H^*(BG, \mathbb{Q}) = \mathbb{Q}[\lambda]$ — becomes then a nontrivial combinatorial task.

2. Graph spaces and universal line bundles. Let X be a compact projective manifold. For $d \in H_2(X)$ we call the *graph space* and denote by

$G_d(X)$ the moduli space of genus 0 holomorphic maps to $\mathbb{C}P^1 \times X$ of degree $(1, d)$. This space can be considered as a compactification of the space of degree d holomorphic maps $\mathbb{C}P^1 \to X$. The automorphism group of $\mathbb{C}P^1$ acts naturally on $G_d(X)$.

Consider first the case $X = \mathbb{C}P^{N-1} = (\mathbb{C}^N - 0)/(\mathbb{C} - 0)$. Denote by $L_d, d \in \mathbb{Z}_+$, the projectivization $Proj(\mathbb{C}^N \otimes S^d(\mathbb{C}^2))$ of the space of degree d vector binary forms $(P_1(\zeta : \xi) : ... : P_N(\zeta : \xi))$. It is a projective space of dimension $Nd + N - 1$ and carries a natural action of $Aut(\mathbb{C}P^1) = PSL_2(\mathbb{C})$ too.

We define the $Aut(\mathbb{C}P^1)$-equivariant map $\mu : G_d(\mathbb{C}P^{N-1}) \to L_d$. Consider a stable degree $(1, d)$ map $\psi : C \to \mathbb{C}P^1 \times \mathbb{C}P^{N-1}$. There exist a unique irreducible component $C_0 \in C$ such that $\psi|_{C_0}$ has degree $(1, d')$ where $d' \leq d$. The image $\psi(C_0)$ is the graph of a map $\mathbb{C}P^1 \to \mathbb{C}P^{N-1}$ of degree d'. The map is given by the binary forms $(P_1 : ... : P_N)$ of degree d' with no common factors and determines the forms uniquely up to a non-zero constant factor. The curve $C - C_0$ has r connected components which are mapped to $\mathbb{C}P^1 \times \mathbb{C}P^{N-1}$ with degrees $(0, d_1), ..., (0, d_r)$, $d_1 + ... + d_r = d - d'$, and the image of i-th component is situated in the slice $(a_i : b_i) \times \mathbb{C}P^{N-1}$. We put $\mu([\psi]) = \Pi_{i=1}^r (a_i \xi - b_i \zeta)^{d_i}(P_1 : ... : P_r)$.

Proposition 2.1 (see the Main Lemma in [10]).
The $Aut(\mathbb{C}P^1)$-equivariant map $\mu : G_d(\mathbb{C}P^{N-1}) \to L_d$ is regular.

For a compact projective submanifold $X \subset \mathbb{C}P^{N-1}$ the graph space $G_d(X)$ is embedded into the space $G_D(\mathbb{C}P^{N-1})$ where $D = p(d)$ is the intersection index of the class $p \in H^2(X)$ of hyperplane sections with the fundamental class of degree d curves. Consider the hyperplane line bundle over the projective space L_D and induce it to $G_d(X)$ by the restriction of the map μ. We will call it the *universal* line bundle corresponding to the embedding of X to $\mathbb{C}P^{N-1}$. The universal line bundle is $Aut(\mathbb{C}P^1)$-equivariant.

Let now X be a compact projective manifold provided with a holomorphic action of a compact Lie group G. Let us assume for simplicity that the 1-st Chern class identifies the Picard group of X with $H^2(X)$. Represent a basis in $H^2(X)$ over $\mathbb{Q}$ by the 1-st Chern classes of very ample G-equivariant line bundles over X. The embeddings of X to projective spaces defined by holomorphic sections of these line bundles are G-equivariant. The universal line bundles over $G_d(X)$ corresponding to these embeddings are $Aut(\mathbb{C}P^1) \times G$ -equivariant in this case. We can now extend the construction of universal

line bundles from ample to arbitrary elements of the Picard group by additivity. For a given equivariant line bundle over X with the G-equivariant 1-st Chern class $p \in H^2_G(X)$ we will denote by the same letter p the $Aut(\mathbb{C}P^1) \times G$-equivariant 1-st Chern class of the corresponding universal line bundle over $G_d(X)$. We will call $p \in H^2_{PSL_2(\mathbb{C}) \times G}(G_d(X))$ the *universal* class corresponding to $p \in H^2_G(X)$.

In the sequel we will use S^1-equivariant cohomology with respect to the maximal torus $S^1 \subset PSL_2(\mathbb{C})$. We will use the lifting of this action to $\mathbb{C}^2$ and to the spaces of binary forms defined in the way breaking the $PSL_2(\mathbb{C})$-symmetry:

$$(\zeta, \xi) \mapsto (\exp(2\pi i \phi)\,\zeta,\ \xi).$$

If $-p$ denotes the equivariant 1-st Chern class of the S^1-equivariant Hopf bundle defined by this lifting, then the equivariant integration over $\mathbb{C}P^1$ and L_D is described respectively by the residue formulas

$$f(p, \hbar) \mapsto \frac{1}{2\pi i} \oint \frac{f(p, \hbar)dp}{p(p + \hbar)},$$

$$f(p, \hbar) \mapsto \frac{1}{2\pi i} \oint \frac{f(p, \hbar)dp}{p(p - \hbar)(p - 2\hbar)...(p - D\hbar)}\ ,$$

where $\mathbb{Z}[\hbar] = H^*(\mathbb{C}P^\infty) = H^*(BS^1)$ stands for the coefficient ring of the S^1-equivariant cohomology theory.

Remark. The graph spaces can be considered as approximations to the space of loops in X. In this interpretation the universal classes correspond to S^1-equivariant forms

$$\text{(symplectic structure)}\ +\ \hbar\ \text{(action functional)}$$

on the loop space. Though heuristically important, such a relation with the Floer theory on the loop space (see [8]) is technically avoidable. The same applies to Proposition 2.1 (see the remark after Proposition 4.1).

3. Symplectic toric manifolds. Consider the standard real symplectic space $\mathbb{C}^N$ provided with the symplectic structure $\text{Im} \sum d\bar{z}_j \wedge dz_j/2$. The *maximal* torus T^N acts on $\mathbb{C}^N$ by linear symplectic transformations $z \mapsto diag(\exp 2\pi i x_1, ..., \exp 2\pi i x_N)z$. The momentum map J of this action maps $\mathbb{C}^N$ onto the closed 1-st orthant $\mathbb{R}^N_+$ in $Lie^* T^N$.

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Consider a subtorus $T^k \subset T^N$. The momentum map of the action $T^k :$ $\mathbb{C}^N$ is the composition of J with the linear integral projection $M : \mathbb{R}^N_+ \to$ $\mathbb{R}^k = Lie^* T^k$. Pick a regular value t of the momentum map and define the symplectic orbifold X as the symplectic reduction $X = \mathbb{C}//_t T^k = (M \circ J)^{-1}(t)/T^k$.

Denote by $K \subset \mathbb{R}^k$ the connected component containing t of the regular value set. The orbifold X is canonically identified with the complex quotient $(M \circ J)^{-1}(K)/T^k_{\mathbb{C}}$. This quotient is a projective toric variety of dimension $N - k$. It has no singularities of complex codimension 1.

The symplectic form on X depends on the choice of the momentum value $t \in K$ and is Kähler with respect to the complex structure. The action of the quotient torus T^N/T^k on X preserves the form. The momentum polyhedron of this action is the fiber $M^{-1}(t)$ in the orthant $\mathbb{R}^N_+$. The orbifold X is compact if and only if the polyhedron is compact or, equivalently, if $M^{-1}(0) = \{0\}$. If it is the case, we call X a compact symplectic toric variety.

Vice versa, a compact projective toric variety X with at most abelian quotient singularities can be obtained as the above symplectic reduction $\mathbb{C}^N//_t T^k$ with N equal to the number of hyperplane walls of the the momentum polyhedron. The polyhedron is described by N inequalities $H_j \geq c_j$ where H_j are reduced integral linear functions on $Lie^* T^{N-k}$. These functions define an embedding of the polyhedron onto a section of the 1-st orthant in $\mathbb{R}^N$.

In the sequel we will assume that X is a compact symplectic toric variety $\mathbb{C}//_t T^k$ with the number of hyperplane walls of the momentum polyhedron equal to N. We review below some basic properties of X (see [1, 7]).

(1) The variety X is non-singular if and only if all the k-dimensional faces of the orthant $\mathbb{R}^N_+$ whose projection to $\mathbb{R}^k$ contain K are mapped to $\mathbb{R}^k$ with the determinant ± 1.

(2) The correspondence between the regular values $t \in K$ of the momentum map and cohomology classes of Kähler symplectic forms is linear and extends to an isomorphism $\mathbb{R}^k \to H^2(X, \mathbb{R})$. The isomorphism identifies K with the Kähler cone of X, and the Picard group $H^2(X, \mathbb{Z})$ — with the lattice $\mathbb{Z}^k \subset \mathbb{R}^k$ of characters of the torus T^k. The 1-st Chern class of the tangent sheaf $\mathcal{T}_X$ is represented by the projection $M(u_1 + ... + u_N)$ of the vector $(1, ..., 1) \in \mathbb{R}^N$.

(3) The cohomology algebra $H^*(X, \mathbb{C})$ is multiplicatively generated by Kähler classes and can be described as follows. In the space $Lie T^N_{\mathbb{C}}$ dual

to $\mathbb{R}_{\mathbb{C}}^N$ consider the union X^* of $(N-k)$-dimensional coordinate subspaces — the orthogonal complements to those k-dimensional faces of $\mathbb{R}_+^N$ whose projections to R^k contain K. Denote by $\mathcal{J}$ the ideal of X^* in the algebra $\mathbb{C}[u_1,...,u_N]$ of regular functions on $LieT_{\mathbb{C}}^N$. In the algebra $\mathbb{C}[p_1,...,p_k]$ of regular functions on $LieT_{\mathbb{C}}^k$ consider the ideal $\mathcal{I}$ induced from $\mathcal{J}$ by the embedding $LieT_{\mathbb{C}}^k \subset LieT_{\mathbb{C}}^N$. Then $H^*(X,\mathbb{C})$ is canonically isomorphic to $\mathbb{C}[p]/\mathcal{I}$. The isomorphism is induced by the correspondence between the infinitesimal characters u_j and 1-st Chern classes of invertible sheaves on X. This description is valid over $\mathbb{Q}$, and for non-singular X — over $\mathbb{Z}$.

(4) The algebra $\mathbb{C}[u]/\mathcal{J}$ of regular functions on X^* is similarly identified with the *equivariant* cohomology algebra of X with respect to the action of the quotient torus T^N/T^k. By definition, the equivariant cohomology algebra of a G-space Y is the cohomology algebra of the homotopic quotient $EG \times_G Y$ and has the module structure over the ring $H^*(BG)$ of G-characteristic classes. In the case of a torus G the ring is identified with the polynomial algebra on $LieG$. The $\mathbb{C}[LieT^N/T^k]$-module structure on $\mathbb{C}[X^*]$ is defined by the projection of X^* along $LieT^k$.

We will use the T^N-equivariant cohomology algebra $H_{T^N}^*(X,\mathbb{C})$. Denote by $\mathbb{C}[\lambda_1,...,\lambda_N]$ another copy of the characteristic class algebra $H^*(BT^N,\mathbb{C})$ which will play the role of the coefficient ring of T^N-equivariant theory throughout the paper. Denote by $(m_{ij})_{i=1}^k {}_{j=1}^N$ the matrix of the projection $M : \mathbb{R}_+^N \to \mathbb{R}^k$. The $\mathbb{C}[\lambda]$-algebra $H_{T^N}^*(X,\mathbb{C})$ is multiplicatively generated by $(p_1,...,p_k)$ satisfying the relations $u_j = \sum_{i=1}^k p_i m_{ij} - \lambda_j$ where $(u_1,...,u_N)$ are the generators of $\mathbb{C}[X^*] = \mathbb{C}[u]/\mathcal{J}$. For example, if $M = (1,...,1)$ then $X = \mathbb{C}P^{N-1}$, $\mathcal{J} = (u_1...u_N)$, and $H_{T^N}^*(X,\mathbb{C})$ is isomorphic to $\mathbb{C}[p,\lambda]/((p-\lambda_1)...(p-\lambda_N))$. Here $-p$ represents the equivariant 1-st Chern class of the Hopf bundle over $\mathbb{C}P^N$ provided with a natural lift of the torus action. In general equivariant 1-st Chern classes of T^N-equivariant invertible sheaves on X are represented by integral linear combinations of u_j. In particular, $u_1 + ... + u_N$ represents the equivariant anti-canonical class of X.

(5) Integration over the fundamental cycle defines an $H^*(BG)$-linear functional with values in $H^*(BG)$ and the corresponding $H^*(BG)$-bilinear Poincare pairing $\langle f,g \rangle = \int_X f \wedge g$ on the equivariant cohomology algebra of a compact G-manifold. The Borel fixed point localization theorem gives rize to the following explicit description of such integration in the case of the

toric symplectic variety X:

$$\int_X f(p, \lambda) = \sum_\alpha \mathrm{Res}_\alpha \frac{f(p, \lambda)\ dp_1 \wedge ... \wedge dp_k}{u_1(p, \lambda)...u_N(p, \lambda)} \ .$$

Here the index α runs the set of fixed points of the torus action on X. The fixed points correspond to the vertices of the momentum polyhedron $M^{-1}(t)$ and therefore — to those k-faces of $\mathbb{R}_+^N$ whose projections to $\mathbb{R}^k$ contain t. The orthogonal complement of such a face is given by k equations $u_{j_1} = ... = u_{j_k} = 0$, $j_1 < ... < j_k$. The symbol Res_α refers to the residue of the k-form at the pole specified by the ordered set of equations

$$\sum_{i=1}^{k} p_i m_{ij_s} = \lambda_s, \ \ s = 1, ..., k.$$

Permutations of the equations affect the sign of the residue.

We will denote $p(\alpha) = (p_1(\alpha), ..., p_k(\alpha))$ the solution to this system of linear equations and $u_j(\alpha)$ the values of the linear functions $u_j = \sum p_i m_{ij} - \lambda_j$ at the point $p(\alpha)$. We will also identify in our notations the index α of the fixed point of the T^N-action on X with the corresponding vertex of the momentum polyhedron and with the set $\{j_1, ..., j_k\}$.

Although the residues on the RHS are rational functions of λ, their sum is a polynomial provided that $f \in H_{T^N}^*(X, \mathbb{C})$ is represented by a polynomial $f(p, \lambda)$. The equivariant Poincare pairing $\langle f, g \rangle = \int_X fg$ is non-degenerate over $\mathbb{Q}[\lambda]$ (and even $\mathbb{Z}[\lambda]$ if X is non-singular). The value at $\lambda = 0$ of the LHS represents $\int_X f(p, 0)$ and gives rise to the ordinary Poincare pairing on $H^*(X)$.

Notice that the toric variety X and its properties have been described entirely in terms of the N integer vectors $Mu_1, ..., Mu_N$ in $\mathbb{R}^k$ and the chosen Kähler cone $K \subset \mathbb{R}^k$.

We will assume further on that X denotes a *non-singular* compact symplectic toric manifold.

A codimension l complete intersection $Y \subset X$ is by definition the common zero locus of holomorphic sections of l line bundles. Let $v_1, ..., v_l$ be the infinitesimal characters of T^N which specify l equivariant line bundles $\mathcal{L}_a$. We will assume that $\mathcal{L}_a$ are non-negative, i.e. $Mv_a \in \bar{K}$, and therefore that the bundle $\mathcal{V} = \oplus_a \mathcal{L}_a$ is convex.

We provide the bundle $\mathcal{V}$ with the additional action of the l-dimensional torus T^l acting fiberwise by scalar multiplication in each summand $\mathcal{L}_a$. We will study the equivariant Gromov - Witten theory of the pair $(X, \mathcal{V})$ with respect to the action of the torus $T = T^N \times T^l$. The algebra

$$\mathbb{Q}[\lambda_1, ..., \lambda_N, \lambda'_1, ..., \lambda'_l] = H^*(BT, \mathbb{Q})$$

will play the role of the coefficient algebra of the theory. Now on we will assume that u_j, v_a, p_i denote corresponding T-equivariant cohomology classes. In particular the T-equivariant 1-st Chern classes of the line bundles $\mathcal{L}_a$ and their fixed point localizations are represented by the linear combinations $v_a := \sum l_{ia} p_i - \lambda'_a$, and $v_a(\alpha) = \sum l_{ia} p_i(\alpha) - \lambda'_a$.

We introduce the T-equivariant integration over the *virtual* fundamental class $[Y]$ of the corresponding (invariant) complete intersections:

$$\int_{[Y]} f \ := \ \int_X f \ Euler(\mathcal{V}) = \sum_\alpha \mathrm{Res}_\alpha \ f(p, \lambda, \lambda') \ \frac{v_1...v_l \ dp_1 \wedge ... \wedge dp_k}{u_1...u_N}.$$

In the non-equivariant limit $\lambda = 0, \lambda' = 0$ it reduces to the integration over the fundamental class of non-singular comlete intersections $Y \subset X$ as well as the whole Gromov-Witten theory — to that for Y, according to the axiom (7).

Degrees of compact holomorphic curves in X form the semigroup

$$\Lambda = \{d \in H_2(X) | (t, d) \geq 0 \ \forall t \in \bar{K}\}.$$

The Kähler cone $\bar{K}$ is the intersection $\cap_\alpha \Delta_\alpha$ of images in $\mathbb{R}^k$ of k-faces in $\mathbb{R}^N_+$. Respectively, Λ coincides with the convex hull of the union $\cup_\alpha \Delta^*_\alpha$ of the orthants $\Delta^*_\alpha \subset \mathbb{R}^{k*}$ polar to the orthants Δ_α. We will denote by $\mathbb{Z}[[\Lambda]]$ the formal completion of the semigroup ring $\mathbb{Z}[\Lambda]$ and represent an element $d \in \Lambda$ by the monomial $q^d = q_1^{d_1}...q_k^{d_k} \in \mathbb{Z}[\Lambda]$ where $(d_1, ..., d_k)$ are coordinates of d in $(\mathbb{Z}^k)^*$.

We will write $d \geq d'$ if $d - d' \in \Lambda$. It is a partial order on $H_2(X)$.

We will also use the notations

$$D_j = \sum_i d_i m_{ij}, \ L_a = \sum_i l_{ia} d_i$$

(which mask the actual dependence of these integers on the vector d).

We complete this section with some notations and elementary information about the 1-dimensional orbits of $T_{\mathbb{C}}^N : X$ which will be exploited, in accordance with the axiom (8), in the fixed point localization technique.

The 1-dimensional orbits correspond to the 1-dimensional edges of the momentum polyhedron. The vertex α of the momentum polyhedron $M^{-1}(t)$ is situated in the k-face $\{(T_1, ..., T_N)|T_s = 0, \forall s \notin \alpha\}$ of $\mathbb{R}_+^N$ and connected by one-dimensional edges to $N - k$ other vertices $\beta(\alpha, j), j \notin \alpha$, situated in the $(k + 1)$-faces $\{(T_1, ..., T_N)|T_s = 0 \forall s \notin \alpha, s \neq j\}$. The edge itself is the momentum polyhedron of the closure of a 1-dimensional toric orbit $\mathbb{C}^{k+1}//_t T^k$ in X isomorphic to $\mathbb{C}P^1$. We denote by $d(\alpha, j) \in \Lambda$ the degree of this $\mathbb{C}P^1$ in X. The corresponding $D_s(\alpha, j) := \sum d_i(\alpha, j)m_{is}$ vanish for $k - 1$ values of the index $s \in \alpha \cap \beta(\alpha, j)$ and are equal to 1 for the two values of $s \in (\alpha \triangle \beta) = \{j, j'\}$. We denote by $L_a(\alpha, j)$ the corresponding values $\sum d_i(\alpha, j)l_{ia}$ of L_a. Notice that β and $j' \in \alpha$ depend on and are uniquely determined by a choice of α and $j \notin \alpha$.

For $\beta = \beta(\alpha, j)$ we have $d(\alpha, j) = d(\beta, j')$ and

$$u_s(\alpha) = u_s(\beta) + D_s(\alpha, j)u_j(\alpha), \ \ v_a(\alpha) = v_a(\beta) + L_a(\alpha, j)u_j(\alpha),$$

In particular, $u_j(\alpha) = -u_{j'}(\beta)$.

4. The generating function. Let $(X, \mathcal{V})$ be the nonsingular compact toric symplectic manifold provided with the convex bundle equivariant with respect to the action of $T = T^N \times T^l$ as described in the previous section. Denote by $\mathcal{V}_d$ the vector bundle over $G_d(X)$ with the fiber $H^0(C, \psi^*\pi_2^*\mathcal{V})$ over the point represented by the stable map $\phi : C \to \mathbb{C}P^1 \times X$. We will study the following series:

$$\mathcal{G}(z, q, \hbar; \lambda, \lambda') = \sum_{d \in \Lambda} q^d \int_{[G_d(X)]} e^{P_1 z_1 + ... + P_k z_k} Euler(\mathcal{V}_d).$$

Here $(P_1, ..., P_k)$ denote the universal classes in $H^2_{T \times S^1}(G_d(X), \mathbb{Q})$ corresponding to the T-equivariant classes $p_i \in H^2_T(X)$.

Proposition 4.1. *The series* $\mathcal{G} = \sum g_{d,m}q^d z^m$ *has polynomial coefficients* $g_{d,m} \in \mathbb{Q}[\hbar, \lambda, \lambda']$.

Proof. The equivariant cohomology classes P_i and $Euler(\mathcal{V}_d)$ of $G_d(X)$ are defined over $\mathbb{Q}[\hbar, \lambda, \lambda']$, and the integration over the Gromov-Witten class $[G_d(X)]$ assumes values in $\mathbb{Q}[\hbar, \lambda, \lambda']$ by the axiom (1).

Remark. The polynomiality property of $g_{d,m}$ is crucial for our proof of Theorems 0.1, 0.2. However one can define the series $\mathcal{G}$ in terms of Gromov-Witten theory on $\mathbb{C}P^1 \times X$ (see [10] , Section 6) without mentioning the universal classes P_i, and the polynomial property then follows directly from the axiom (1). Thus our use of Proposition 2.1 is avoidable.

Proposition 4.2. (see [10]).

$$\mathcal{G}(z,q,\hbar) = \int_{[Y]} \mathcal{S}(q\exp(\hbar z),\hbar) \; e^{pz} \; \mathcal{S}(q,-\hbar) \; ,$$

where $\mathcal{S}(q,\hbar;\lambda,\lambda')$ is determined from the condition that for any $a \in H_T^*(X)$

$$\int_{[X]} \mathcal{S} \; a = \int_{[X]} Euler(\mathcal{V}) \; a + \sum_{d\in\Lambda-0} q^d \int_{[X_{2,d}]} \frac{\mathrm{ev}_1^*(a) \; Euler(\mathcal{V}_{2,d})}{(\hbar-c)} \; .$$

Proof. The proposition is easily deduced (see [10]) by localization to fixed points of the S^1-action on $G_d(X)$ and L_D and from the following corollary of the axiom (2):

$$\int_{[X_{2,d}]} \frac{\mathrm{fgt}_2^*(A)}{\hbar-c} = \int_{[X_{1,d}]} \frac{A}{\hbar(\hbar-c)}.$$

Corollary 4.3. $\mathcal{S} = 1 + o(1/\hbar)$ *when written as a formal power series in* $\hbar^{-1}$.

Proof: $Euler(\mathcal{V}_{2,d}) = \mathrm{fgt}_2^*(Euler(\mathcal{V}_{1,d}))$.

Remark. The vector-function $J_{\mathcal{V}}$ in Theorem 0.2 is defined in Section 1 as $\mathcal{S} \; \exp((t_0 + \sum P_i \log q_i)/\hbar)$.

Denote by $\mathcal{S}_\alpha$ the restriction of the T-equivariant cohomology class $\mathcal{S} = \sum_{d\in\Lambda} S_{\alpha,d} q^d$ to the fixed point α of the torus T action on X. The coefficients $S_{\alpha,d}$ of the series $\mathcal{S}_\alpha$ are rational functions of $(\hbar,\lambda,\lambda')$.

Proposition 4.4. *The series* $\mathcal{S}_\alpha$ *satisfy the following recursion relations:*

$$\mathcal{S}_\alpha(q,\hbar;\lambda,\lambda') = \sum_{d\in\Lambda} R_{\alpha,d}(\hbar^{-1};\lambda,\lambda')q^d +$$

$$\sum_{j\notin\alpha} \sum_{n=1}^{\infty} q^{n \; d_{\alpha,j}} \frac{C_{\alpha,j}(n)}{n\hbar + u_j(\alpha)} \mathcal{S}_{\beta(\alpha,j)}(q,-u_j(\alpha)/n;\lambda,\lambda') \; ,$$

where $R_{\alpha,d}$ are polynomials in $\hbar^{-1}$ with coefficients in $\mathbb{Q}(\lambda, \lambda')$, and the recursion coefficients $C_{\alpha,j}(n) =$

$$\frac{\Pi_a \Pi_{m=1}^{nL_a(\alpha,j)} (v_a(\alpha) - mu_j(\alpha)/n) \; \Pi_{s \notin \alpha \cup \beta} \Pi_{m \leq 0} (u_s(\alpha) - mu_j(\alpha)/n))}{(n-1)!(u_j(\alpha)/n)^{n-1}) \; \Pi_{s \notin \beta} \Pi_{m \leq nD_s(\alpha,j)} (u_s(\alpha) - mu_j(\alpha)/n)} \; .$$

Proof. We apply the fixed point localization technique to the action of the torus T on $X_{2,d}$. Consider a fixed point represented by the stable map $f : (C, x_1, x_2) \to X$ and denote by C' the irreducible component of C carrying the marked point x_1. The contribution of this fixed point to $\mathcal{S}_\alpha$ via the localization formula described in the axiom (8) vanishes unless $f(x_1) = \alpha$.

If $f(C') = \alpha$, then the contribution involves integration over the space $\bar{\mathfrak{M}}_r$ of configurations of $r > 2$ special points on the connected component of $f^{-1}(\alpha)$ containing x_1. The universal cotangent line at the 1-st marked point localizes to the line bundle over $\bar{\mathfrak{M}}_r$ formed by the cotangent lines to C' at x_1. Since the action of T on this line bundle is trivial, the localization of the Chern class c is nilpotent. Thus the contribution of such fixed points to the localization formula is polynomial in $1/\hbar$.

If $f(x_1) = \alpha$ but $f(C') \neq \alpha$ then $f' := (f|C')$ is the degree n cover of a 1-dimensional orbit of $T_\mathbb{C}$ in X connecting the fixed point α with another fixed point $\beta = \beta(\alpha, j)$. In this case f is glued from $f' : (C', x_1, x) \to X$ and a stable map $f'' : (C'', x, x_2) \to X$ of degree $d'' = d - nd(\alpha, j)$ with $f'(x) = f''(x) = \beta$. The localization of $\hbar - c$ in this case equals $\hbar + u_j(\alpha)/n$. The contribution of $T'_x \otimes T''_x$ to the Virtual Normal Euler Class in the localization axiom equals $-u_j(\alpha)/n - c$ where c is the equivariant 1-st Chern class of the universal cotangent line over $X_{2,d''}$ at the marked point x. The remaining part of the Virtual Normal Euler Class is the product of such classes for f'' in $X_{2,d''}$ and for f' in $X_{2,nd(\alpha,j)}$. The latter one can be easily computed by using the quotient description of the tangent bundle to $X = \mathbb{C}^N //T^k$ as a virtual direct sum of line bundles and is equal to $1/C_{\alpha,j}(n)$. Also we have $|Aut f| = n|Aut f''|$ where n is the order of the cyclic automorphism group of the n-fold cover $f' : z \mapsto z^n$.

The summation over all d'', $j \notin \alpha$ and $n = 1, 2, ...$ gives rise to the recursion relation. $\square$

Proposition 4.5. *The series $\mathcal{S}$ is uniquely determined by the following properties:*
(a) the recursion relations of Proposition 4.4,

(b) the asymptotic condition $\mathcal{S} = 1 + o(1/\hbar)$ of Corollary 4.3,
(c) the property from Proposition 4.1 of the series

$$\sum g_{d,m} q^d z^m = \int_{[Y]} \mathcal{S}(q e^{\hbar z}, \hbar) e^{pz} \mathcal{S}(q, -\hbar)$$

to have polynomial coefficients $g_{d,m} \in \mathbb{Q}[\hbar, \lambda, \lambda']$.

Proof. Consider another solution $\mathcal{S}'$ to the same recursion relations (a) with the same polynomials $R'_{\alpha,d} = R_{\alpha,d}$ as in $\mathcal{S}$ for all $0 \le d < d_0$ and satisfying the asymptotic condition (b). We will show that $R'_{\alpha,d_0} = R_{\alpha,d_0}$. This would imply the uniqueness since, given all the polynomials $R_{a,d}$, the recursion relations (a) allow one to recover the series $\mathcal{S}$ unambiguously.

Consider the q^{d_0}-term in $\mathcal{G}' - \mathcal{G}$. The conditions (a),(b) imply that $\mathcal{S}' - \mathcal{S} = Rq^{d_0} +$ (higher order terms in q) and that q^{d_0}-term in $\mathcal{G}' - \mathcal{G}$ is equal to

$$\delta(R) = \int_{[Y]} e^{(p + d_0 \hbar)z} R(\hbar, \lambda) + e^{pz} R(-\hbar, \lambda) \ ,$$

where the class R is defined by its localizations $R_\alpha = R'_{\alpha,d_0} - R_{\alpha,d_0}$. From (b) and (c) we know that R is a polynomial in $1/\hbar$ divisible by $1/\hbar^2$ and that the series $\delta = \sum \delta_m z^m$ has polynomial coefficients $\delta_m \in \mathbb{Q}[\hbar, \lambda, \lambda']$.

A generic value of $\lambda = (\lambda_1, ..., \lambda_N)$ determines a linear function on the momentum polyhedron $\mathbb{R}^N_+ \cap J^{-1}(z)$ for each $z \in K$ with pairwise distinct values H_α at the vertices. Computing $\delta(A\hbar^{-1} + B)$ modulo $\hbar$ for a generic ray $t \mapsto tz$ we find

$$\sum_\alpha e^{H_\alpha t} (A_\alpha t \ (z, d_0) + B_\alpha) \operatorname{Res}_\alpha \frac{v_1 ... v_l \ dp_1 \wedge ... \wedge dp_k}{u_1 ... u_N}.$$

Here $v_1(\alpha) ... v_a(\alpha) \ne 0$ for generic values of λ', and $(z, d_0) > 0$. Since the functions $\exp(H_\alpha t), t \exp(H_\alpha t)$ with distinct $H\alpha$ are linearly independent, we conclude that $\delta(A/\hbar + B) = o(\hbar)$ implies $A = B = 0$ in $H^*_T(X, \mathbb{C}(\lambda, \lambda'))$ for generic and therefore for all values of λ, λ' and z. Applying this conclusion to $\hbar^{2r} R(1/\hbar, \lambda)$ with $r > 0$ and $R(1/\hbar) = A\hbar^{-2r-1} + B\hbar^{-2r} + ...$ of degree $\le (2r+1)$ we find that $\deg R \le (2r-1)$ and by induction — that $\deg R \le 1$. Now the assumption (b) that R has no terms of order $\hbar^0$ and $\hbar^{-1}$ implies that $R = 0$. $\square$

5. Toric map spaces. In this section we describe toric compactifications of spaces of holomorphic maps $\mathbb{C}P^1 \to X$.

For each $d \in \Lambda$ consider the following complex space of N-dimensional vector-polynomials in one complex variable ζ:

$$\{(z_1(\zeta), ..., z_N(\zeta))| \deg z_j \le D_j\}$$

The torus T^N acts on this space componentwise. Denote J_d the momentum map of the induced action of $T^k \subset T^N$, pick $t \in K$ and introduce the symplectic toric variety $X_d = J_d^{-1}(t)/T^k$. It is compact and nonsingular whenever X is compact and nonsingular. Generic points in X_d represent degree d holomorphic maps

$$\mathbb{C}P^1 \to X : \zeta \mapsto (z_1(\zeta), ..., z_N(\zeta)) \bmod T_{\mathbb{C}}^k .$$

The variety X_d is empty unless $d \in \cup_\alpha \Delta_\alpha^*$.

The rotation $\zeta \mapsto \exp(2\pi i \phi)\zeta$ of $\mathbb{C}P^1$ induces an S^1-action on X_d. Fixed points of the $S^1 \times T^N$-action on X_d are isolated. The Borel localization formula for X_d yields:

$$\int_{X_d} f(p, \hbar, \lambda) = \sum_{\alpha:d\in\Delta_\alpha^*} \sum_r \text{Res}_{\alpha,r} \frac{f(p, \hbar, \lambda)dp_1 \wedge ... \wedge dp_k}{\Pi_{j:D_j \ge 0}(u_j(u_j - \hbar)(u_j - 2\hbar)...(u_j - D_j\hbar))} .$$

Here $\text{Res}_{\alpha,r}$ refers to the residue at the point specified by the equations

$$u_{j_1}(p, \lambda) = r_1\hbar , ..., \ u_{j_k}(p, \lambda) = r_k\hbar , j_1 < ... < j_k,$$

with $\{j_1, ..., j_k\} = \alpha$ and r runs the integer vectors

$$r = (r_1, ..., r_k) : 0 \le r_1 \le D_{j_1}, ..., 0 \le r_k \le D_{j_k} .$$

The equivariant cohomology algebra of X_d is identified with the quotient of $\mathbb{C}[p, \hbar, \lambda]$ by the kernel ideal of the corresponding Poincaré pairing.

The dimension of X_d may exceed the Riemann-Roch dimension $\dim X + (c_1(\mathcal{T}_X), d) = N - k + D_1 + ... + D_N$ of the space of degree d holomorphic maps $\mathbb{C}P^1 \to X$ if some $D_j = \sum d_i m_{ij}$ are negative. We introduce the *virtual fundamental class* $[X_d]$ of the Riemann-Roch dimension as the Poincaré-dual to the equivariant Euler class of the following vector bundle over X_d.

Consider the complex space of the vector polynomials in the variable ζ^{-1} defined by

$$\{(z_1(\zeta^{-1}), ..., z_N(\zeta^{-1}))|z_j(0) = 0, \deg z_j < -D_j\} .$$

We introduce the T^N-equivariant locally free sheaf on $X_d = J_d^{-1}(t)/T^k$ associated with the componentwise action of $T^k \subset T^N$ on this space. The rotation of ζ defines an S^1-equivariant structure on the sheaf. At generic points $\psi : \mathbb{C}P^1 \to X$ of X_d the sheaf coincides with the obstruction sheaf $H^1(\mathbb{C}P^1, \psi^*(\mathcal{T}_X))$.

The Borel localization formula gives rise to the following description of integration over the virtual fundamental class:

$$\int_{[X_d]} f = \sum_{\alpha:d\in\Delta_\alpha^*} \sum_r \operatorname{Res}_{\alpha,r} dp_1 \wedge ... \wedge dp_k \ f(p,\hbar,\lambda)\Pi_{j=1}^N \frac{\Pi_{m=-\infty}^{-1}(u_j - m\hbar)}{\Pi_{m=-\infty}^{D_j}(u_j - m\hbar)} \ .$$

Consider now the convex bundle $\mathcal{V} = \oplus_a \mathcal{L}_a$. For a generic degree-$d$ map $\psi : \mathbb{C}P^1 \to X$ the space $H^0(\mathbb{C}P^1, \psi^*\mathcal{L}_a)$ can be identified with the space of polynomials $\{y(\zeta)| \deg y \le L_a\}$ where $L_a = \sum d_i l_{ia}$. We introduce the vector bundle of dimension $L_a + 1$ over X_d associated with the scalar action of T^k on this space via the character $\sum l_{ia} p_i$. The direct sum of these bundles over $a = 1, ..., l$ is equivariant with respect to $S^1 \times T^N \times T^l$ where S^1 acts by rotations of ζ, and T^l acts by scalr multiplication componentwise on the direct summands. The $S^1 \times T$-equivariant Euler class of this bundle is $\Pi_a \ v_a(v_a - \hbar)...(v_a - L_a\hbar)$ where $v_a = \sum l_{ia}p_i - \lambda_a'$. We define the *virtual fundamental class* $[Y_d]$ as the Poincaré-dual to this Euler class:

$$\int_{[Y_d]} f := \int_{[X_d]} f(p,\hbar,\lambda,\lambda')\Pi_a \Pi_{m=0}^{L_a}(v_a(p,\lambda) - m\hbar) \ .$$

For polynomial $f \in \mathbb{Q}[p,\hbar,\lambda,\lambda']$ the integration over $[Y_d]$ assumes polynomial values in $\mathbb{Q}[\hbar,\lambda,\lambda']$.

6. The hypergeometric series. We will study here the generating function

$$\Phi(z,q) = \sum_{d\in\Lambda} q^d \int_{[Y_d]} e^{p_1 z_1 + ... + p_k z_k}$$

which is a formal power series in z with coefficients in the formal completion of the semigroup ring of Λ over the polynomial algebra $\mathbb{Q}[\hbar,\lambda,\lambda']$.

Proposition 6.1. *The series Φ is weighted-homogeneous of degree $l + k - N$ with respect to the grading*

$$\deg z_i = -1, \deg \hbar = \deg \lambda_j = \deg \lambda_a' = 1, \deg q_i = \sum_j m_{ij} - \sum_a l_{ia}.$$

Proof: This follows from grading in equivariant cohomology and the formula $N - k - l + (c_1(\mathcal{T}_X), d) - (c_1(\mathcal{V}), d)$ for the complex dimension of the virtual fundamental cycle $[Y_d]$.

Consider the formal q-series Ψ (which differs from $I_\mathcal{V}$ in Theorem 0.2 by the exponential factor $\exp((t_0 + p \log q)/\hbar)$ only):

$$\Psi(q, \hbar; \lambda, \lambda') = \sum_{d \in \Lambda} q^d \frac{\Pi_a \Pi_{m \leq L_a}(v_a + m\hbar) \Pi_j \Pi_{m \leq 0}(u_j + m\hbar)}{\Pi_a \Pi_{m \leq 0}(v_a + m\hbar) \Pi_j \Pi_{m \leq D_j}(u_j + m\hbar)} \ ,$$

where $u_j = \sum p_i m_{ij} - \lambda_j$ and $v_a = \sum p_i l_{ia} - \lambda'_a$ are T-equivariant cohomology classes of X, the integers $D_j = \sum d_i m_{ij}$, $L_a = \sum d_i l_{ia}$ depend on $d = (d_1, ..., d_k)$, and m runs integer values starting from $-\infty$. Coefficients of the series Ψ are well-defined equivariant cohomology classes of X over the field $\mathbb{C}(\hbar, \lambda)$ of rational functions, and the whole series can be considered as an equivariant cohomology class of X with appropriate coefficients. Notice that a q^d-term in Ψ has zero localization at the fixed point α unless $d \in \Delta_\alpha^*$.

Proposition 6.2 (compare with [8, 9, 10]).

$$\int_{[Y]} \Psi(q \exp(\hbar z), \hbar; \lambda, \lambda') e^{pz} \Psi(q, -\hbar; \lambda, \lambda') = \Phi(z, q, \hbar, \lambda, \lambda').$$

In particular, $\Phi((t - \tau)/\hbar, e^\tau, \hbar)$ is invariant under the change $(t, \tau, \hbar) \mapsto (\tau, t, -\hbar)$.

Proof. We have

$$\int_{[Y]} \Psi(q \exp(\hbar z), \hbar) e^{pz} \Psi(q, -\hbar) =$$

$$\sum_\alpha \sum_{d', d'' \in \Delta_\alpha^*} q^{d' + d''} \operatorname{Res}_\alpha e^{(p + \hbar d')z} \, dp_1 \wedge ... \wedge dp_k \times$$

$$\frac{\Pi_a \Pi_{m = -L_a''}^{L_a'}(v_a + m\hbar) \ \Pi_j[\Pi_{m \leq 0}(u_j + m\hbar) \ \Pi_{m \geq 0}(u_j + m\hbar)]}{\Pi_j u_j[\Pi_{m \leq D_j'}(u_j + m\hbar) \ \Pi_{m \geq -D_j''}(u_j + m\hbar)]}$$

$$= \sum_\alpha \sum_{d \in \Delta_\alpha^*} q^d \sum_r \operatorname{Res}_{\alpha, r} e^{pz} dp_1 \wedge ... \wedge dp_k \times$$

$$\frac{\Pi_a \Pi_{m = -L_a}^{0}(v_a + m\hbar) \ \Pi_j u_j \Pi_{m \in \mathbb{Z}}(u_j + m\hbar)}{\Pi_j u_j[\Pi_{m \leq 0}(u_j + m\hbar) \ \Pi_{m \geq -D_j}(u_j + m\hbar)]}$$

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$$= \sum_{\alpha} \sum_{d \in \Delta_\alpha^*} \sum_r \mathrm{Res}_{\alpha,r} \frac{e^{pz} dp_1 \wedge ... \wedge dp_k \ \Pi_a \Pi_{m=0}^{L_a}(v_a - m\hbar) \ \Pi_j \Pi_{m<0}(u_j - m\hbar)}{\Pi_j \Pi_{m \leq D_j}(u_j - m\hbar)}$$

$$= \sum_{d \in \Lambda} q^d \int_{[Y_d]} e^{pz}.$$

□

Consider the localizations $\Psi_\alpha(q, \hbar; \lambda, \lambda')$ of the class Ψ at the fixed points:

$$\Psi_\alpha = \sum_{d \in \Delta_\alpha^*} q^d \frac{\Pi_a \Pi_{m=1}^{L_a}(v_a(\alpha) + m\hbar)}{\Pi_{j \in \alpha} D_j! \hbar^{D_j}} \ \Pi_{j \notin \alpha} \frac{\Pi_{m \leq 0}(u_j(\alpha) + m\hbar)}{\Pi_{m \leq D_j}(u_j(\alpha) + m\hbar)}.$$

Each q^d-term in this series is a rational function of $\hbar$ and can be uniquely written as the sum of simple fractions with poles at $\hbar = -u_j(\alpha)/n$, $j \notin \alpha$, $0 < n \leq D_j$ plus a Laurent polynomial in $\hbar$.

Proposition 6.3. *The series Ψ_α satisfy the following recursion relations:*

$$\Psi_\alpha(q, \hbar; \lambda, \lambda') = \sum_{d \in \Delta_\alpha^*} q^d R_{\alpha,d}(\hbar, \lambda, \lambda') +$$

$$\sum_{j \notin \alpha} \sum_{n>0} q^{n \ d(\alpha,j)} \frac{C_{\alpha,j}(n)}{(u_j(\alpha) + n\hbar)} \Psi_\beta(q, -u_j(\alpha)/n; \lambda, \lambda') \ ,$$

where $\beta = \beta(\alpha, j)$ and the coefficients $C_{\alpha,j}(n) =$

$$\frac{\Pi_a \Pi_{m=1}^{nL_a(\alpha,j)}(v_a(\alpha) - mu_j(\alpha)/n) \ \Pi_{s \notin \alpha \cup \beta} \Pi_{m \leq 0}(u_s(\alpha) - mu_j(\alpha)/n)}{(n-1)!(u_j(\alpha)/n)^{n-1} \ \Pi_{s \notin \beta} \Pi_{m \leq nD_s(\alpha,j)}(u_s(\alpha) - mu_j(\alpha)/n)} \ .$$

Proof. For generic λ, λ' and $\beta = \beta(\alpha, j)$ denote by Z_β^d the coefficient of the series Ψ_β at q^d. The value of this coefficient at $\hbar = -u_j(\alpha)/n = u_{j'}(\beta)/n$ is equal to

$$\frac{\Pi_a \Pi_{m=1} L_a(v_a(\beta) - mu_j(\alpha)/n) \ \Pi_s \Pi_{m \leq 0}(u_s(\beta) - mu_j(\alpha)/n)}{\Pi_s \Pi_{m \leq D_s}(u_s(\beta) - mu_j(\alpha)/n)}$$

It vanishes unless $d \in \Delta^*(\beta)$ and $D_{j'} + n \geq 0$. Due to the relations $u_s(\beta) = u_s(\alpha) - D_s(\alpha, j)u_j(\alpha)$, $v_a(\beta) = v_a(\alpha) - L_a(\alpha, j)u_j(\alpha)$ we have $Z_\beta^d(-u_j(\alpha)/n) =$

$$\frac{\Pi_a \Pi_{m>nL_a(\alpha,j)}^{L_a+nL_a(\alpha,j)}(v_a(\alpha) - mu_j(\alpha)/n) \ \Pi_s \Pi_{m \leq nD_s(\alpha,j)}(u_s(\alpha) - mu_j(\alpha)/n)}{\Pi_s \Pi_{m \leq D_s+nD_s(\alpha,j)}(u_s(\alpha) - mu_j(\alpha)/n)} \ .$$

Since $D_s(\alpha, j) = 0$ for $s \in \alpha \cap \beta$ and $D_j(\alpha, j) = D_{j'}(\alpha, j) = 1$, we have
$C_{\alpha,j}(n) Z_\beta^d(-u_j(\alpha)/n) =$

$$\frac{\Pi_a \Pi_{m=1}^{L_a + nL_a(\alpha,j)} (v_a(\alpha) - \frac{m}{n} u_j(\alpha)) \ \Pi_{s \neq j,j'} \Pi_{m \leq 0} (u_s(\alpha) - \frac{m}{n} u_j(\alpha))}{\Pi_{m=1}^{D_{j'}+n} (\frac{-n-m}{n} u_j(\alpha)) \ \Pi_{m=1}^{D_j+n} \ _{m \neq n} (\frac{n-m}{n} u_j(\alpha)) \ \Pi_{s \neq j,j'} \Pi_{m \leq D_s + nD_s(\alpha,j)} (u_s(\alpha) - \frac{m}{n} u_j(\alpha))} .$$

The last product is exactly the residue of the $q^{d+nd(\alpha,j)}$-term in the series Ψ_α at the simple fraction with the pole $\hbar = -u_j(\alpha)/n$. The vector $d' = d + nd(\alpha, j)$ is automatically in Δ_α^*, and additionally $D_j + n \geq n$. Vice versa, when d' is in Δ_α^*, and for some $j \notin \alpha$ and $n \in \mathbb{N}$ we have $D_j' \geq n$, then $d = d' - nd(\alpha, j)$ is in $\Delta_{\beta(\alpha,j)}^*$ and $D_{j'} + n = D_{j'}' \geq 0$.

Thus the sum of simple fractions on the RHS of the recursion relations reproduces all the simple fractions of the LHS with non-zero poles, and the remaining part of each $Z_\alpha^{d'}(\hbar)$ is a Laurent polynomial of $\hbar$. Since the identity between the q-series with coefficients in $\mathbb{Q}(\hbar, \lambda, \lambda')$ holds for generic values of (λ, λ') it holds in fact over $\mathbb{Q}(\hbar, \lambda, \lambda')$. $\square$

Remark. The coefficients $C_{\alpha,j}(n)$ in the recurrence relations of Propositions 4.4 and 6.3 are the same. Although we obtained this fact by straightforward computations of both coefficients, the coincidence should not be considered a miracle. The value of the coefficient $C_{\alpha,j}(n)$ in Proposition 4.4 is determined by localizations to fixed points represented by *irreducible* degree n maps $\mathbb{C}P^1 \to X$. On the other hand, the graph spaces differ from the toric map spaces only near stable maps of reducible curves. Perhaps the coincidence of the recurrence coefficients can be derived from this observation.

Consider now the case of complete intersections Y with non-negative 1-st Chern class $c_1(\mathcal{T}_Y)$. This means that

$$\sum_i p_i \left(\sum_j m_{ij} - \sum_a l_{ia} \right) \in \bar{K} .$$

A reformulation of this condition reads: $\deg q^d \geq 0$ for all $d \in \Lambda$.
Put

$$\Lambda_0 = \{ d \in \Lambda \mid \sum L_a = \sum D_j, D_j \geq 0 \forall j = 1, ..., N \},$$

$$\Lambda_1 = \{ d \in \Lambda \mid \sum D_j - \sum L_a = 1, D_j \geq 0 \forall j = 1, ..., N \},$$

$$\Lambda_0' = \{ d \in \Lambda \mid \sum L_a = \sum D_j \}.$$

Proposition 6.4. *Suppose that $\mathcal{T}_Y$ is non-negative. Then*

$$\Psi = \Psi^{(0)} + \Psi^{(1)}/\hbar + o(1/\hbar),$$

$$\Psi^{(0)} = \sum_{d \in \Lambda_0} \frac{L_1! ... L_l!}{D_1! ... D_N!} q^d ,$$

$$\Psi^{(1)} = \sum_{d \in \Lambda_1} \frac{L_1! ... L_l!}{D_1! ... D_N!} q^d + \sum_a g_a(q) v_a - \sum_j f_j(q) u_j,$$

where g_a and f_j are some power series $\sum_{d \in \Lambda_0'} A_d q^d$ with coefficients $A_d \in \mathbb{Q}$ and $A_0 = 0$.

Proof. In the case of non-negative $\mathcal{T}_Y$ we have $\sum D_j \geq \sum L_a$ for all $d \in \Lambda$. The definition of $\Psi(q, \hbar; \lambda, \lambda')$ then shows that it can be written as a power series in $1/\hbar$, and that the terms of order $\hbar^0$ and $\hbar^{-1}$ have the form described in the proposition.

7. Equivalence transformations. We study further the case of complete intersections Y with non-negative tangent sheaf.

Consider a series $Z(q, \hbar; \lambda, \lambda') \in H_T^*(Y, \mathbb{Q})$ of weighted-homogeneous degree 0 with respect to the grading $\deg \hbar = \deg \lambda = \deg \lambda' = 1$, $\deg q_i = \sum_j m_{ij} - \sum_a l_{ia}$ which satisfies
  (i) the recursion relation

$$Z_\alpha(q, \hbar; \lambda, \lambda') = \sum_{d \in \Lambda} R_{\alpha,d}(\hbar^{-1}; \lambda, \lambda') q^d +$$

$$\sum_{j \notin \alpha} q^{n \; d_{\alpha,j}} \frac{C_{\alpha,j}(n)}{n\hbar + u_j(\alpha)} Z_{\beta(\alpha,j)}(q, -u_j(\alpha)/n; \lambda, \lambda') ,$$

where $R_{\alpha,d}$ are polynomials in $\hbar^{-1}$ with coefficients in $\mathbb{Q}(\lambda, \lambda')$ and the recursion coefficients $C_{\alpha,j}(n)$ are described in Propositions 4.4 and 6.3;
  (ii) the condition that the series

$$W = \sum w_{d,m} q^d z^m := \int_{[Y]} Z(q e^{\hbar z}, \hbar) e^{pz} Z(q, -\hbar)$$

has polynomial coefficients $w_{d,m} \in \mathbb{Q}[\hbar, \lambda, \lambda']$.

In Section 4 we proved that the series $\mathcal{S}$ is uniquely determined by these properties and the asymptotic condition of Corollary 4.3. In the previous

section we found a series Ψ which also has these properties but may differ from $\mathcal{S}$ by the asymptotic condition. In this section we describe some transformations of the series Z which preserve the properties (i),(ii) but change the asymptotical expansion $Z = Z^{(0)} + Z^{(1)}/\hbar + o(1/\hbar)$.

Proposition 7.1. *Let $f = \sum_{\Lambda_0'} f_d q^d$ be a series with rational coefficients $f_d \in \mathbb{Q}$ and $f_0 \neq 0$. Then fZ satisfies the conditions (i),(ii).*

Proof. Simultaneous multiplication of the localizations Z_α by f obviously preserves the the recursion relation of Proposition 4.4 and affects the series $\sum R_{\alpha,d} q^d$ in such a way that the polynomiality property of $R_{\alpha,d}(1/\hbar)$ is preserved. This proves (i).

The multiplication of Z by f gives rise to the multiplication of the series W by $f(q \exp(\hbar z)) f(q)$ which obviously preserves the polynomiality property of the coefficients $w_{d,m}$. This proves (ii).

Proposition 7.2. *Let $f = \sum_{d \in \Lambda_0' \cup \Lambda_1} f_d q^d$ be a series of weighted - homogeneous degree 1 where f_d are linear inhomogeneous functions of λ, λ' with rational coefficients, and $f_0 = 0$. Then $\exp(f/\hbar)Z$ satisfies the conditions (i),(ii).*

Proof. We have

$$\exp(-nf/u_j(\alpha) - f/\hbar) = \exp(-(n\hbar + u_j(\alpha)f/u_j(\alpha)\hbar) = 1 + (n\hbar + u_j(\alpha))g_{\alpha,j,n},$$

where $g_{\alpha,j,n}$ is a q-series with coefficients polynomial in $\hbar^{-1}$ and rational in (λ, λ'). This implies that $\exp(f/h)Z_\alpha$ satisfy the recursion relation (i) with the new initial condition

$$e^{f/\hbar} \sum_{d \in \Lambda} R_{d,\alpha} q^d +$$

$$e^{f/\hbar} \sum_{j \notin \alpha} \sum_{n=1}^{\infty} g_{\alpha,j,n} \ q^{n \ d_{\alpha,j}} \ C_{\alpha,j}(n) \ Z_{\beta(\alpha,j)}(q, -u_j(\alpha)/n; \lambda, \lambda')$$

whose coefficients at q^d are still polynomial in $1/\hbar$.

The transformation $Z \mapsto \exp(f/\hbar)Z$ multiplies the series W by $\exp g$, where $g = (f(qe^{\hbar z}) - f(q))/\hbar$ has polynomial coefficients when written as a power series in (q, z) since $\exp(\hbar z d) - 1$ is divisible by $\hbar$. $\square$

Proposition 7.3. *Let the linear combination $fp = f_1 p_1 + ... + f_k p_k$ of the equivariant cohomology classes $p_i \in H_T^*(Y, \mathbb{Q})$ be given by the series*

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$f_i = \sum_{d \in \Lambda_0'} f_{i,d} q^d$ with rational coefficients $f_{i,d} \in \mathbb{Q}$ and $f_{i,0} = 0$. Then $\exp(fp/\hbar)Z(q \exp f, \hbar; \lambda, \lambda')$ satisfies the conditions (i),(ii).

Proof. For $\beta = \beta(\alpha, j)$ we have $p(\alpha) - p(\beta) = d_{\alpha,j} u_j(\alpha)$ and therefore

$$\exp[-fp(\alpha)/\hbar - nfp(\beta)/u_j(\alpha)] =$$

$$e^{n \; d_{\alpha,j} f} \; \exp[-fp(\alpha)(n\hbar + u_j(\alpha))/(u_j(\alpha)\hbar)] = e^{n \; d_{\alpha,j} f} + (n\hbar + u_j(\alpha))g_{\alpha,j,n} \; ,$$

where the series $g_{\alpha,j,n}(q, \hbar^{-1}; \lambda, \lambda')$ has the same properties as specified in the proof of Proposition 7.2. This implies that the recursion relation for $e^{fp/\hbar}Z(qe^f, \hbar)$ (with some initial conditions $\sum R_{\alpha,d} q^d$) follows from the recursion relation obtained by the change $q \mapsto qe^f$ from (i) satisfied by Z.

The operation $Z \mapsto e^{fp}Z(qe^f, \hbar)$ transforms $W(z, q)$ to

$$W\left(z + \frac{f(qe^{\hbar z}) - f(q)}{\hbar}, qe^f\right)$$

and preserves the polynomiality property of the coefficients $w_{d,m}$ since coefficients of the series $f_i(qe^{\hbar z}) - f_i(q)$ are divisible by $\hbar$. $\square$

Using the notations from Proposition 6.4 we perform the following operations with the series Ψ which according to Propositions 7.1, 7.2, 7.3 preserve the properties (i),(ii).

(1) Divide Ψ by $\Psi^{(0)}$; this transforms the asymptotic expansion $\Psi = \Psi^{(0)} + \Psi^{(1)}/\hbar + o(1/\hbar)$ to

$$1 + \hbar^{-1} \left(H(q) + \sum_a G_a(q)v_a - \sum_j F_j(q)u_j\right) + o(\hbar^{-1}),$$

where $H = (\sum_{d \in \Lambda_1} \frac{L_1! ... L_l!}{D_1! ... D_N!} q^d)/\Psi^{(0)}$, and $G_a = g_a/\Psi^{(0)}$, $F_j = f_j/\Psi^{(0)}$.

(2) Divide $\Psi/\Psi^{(0)}$ by $\exp \hbar^{-1}(H + \sum_j F_j \lambda_j - \sum_a G_a \lambda_a')$; this transforms the asymptotic expansion to

$$Z = 1 + \hbar^{-1} \sum_i \phi_i p_i + o(\hbar^{-1}) \; ,$$

where $\phi_i = \sum_a l_{ia} G_a - \sum_j m_{ij} F_j$.

(3) Transform $\exp(-\sum \phi_i p_i/\hbar)Z(q, \hbar)$ to the new variables $Q_i = q_i e^{\phi_i(q)}$, $i = 1, ..., k$; the resulting series $S(Q, \hbar; \lambda, \lambda')$ has the asymptotic expansion $1 + o(1/h)$.

Corollary 7.4. *Suppose that $\mathcal{T}_Y$ is non-negative. Then $\mathcal{S}(q, \hbar; \lambda, \lambda') = S(q, \hbar; \lambda, \lambda')$.*

Proof. The series S satisfies the conditions (a),(b),(c) of Proposition 4.5 which uniquely determine the series $\mathcal{S}$.

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